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Multiple Stable Traveling Wave Profiles of a System of Conservation Laws Arising from Chemotaxis

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Abstract. In this paper, we establish the existence and nonlinear stability of a hyperbolic system of conservation laws derived from a repulsive singular chemotaxis model. By the phase plane analysis alongside Poincaré-Bendixson theorem, we first prove that this hyperbolic system admits three different types of traveling wave profiles, which are explicitly illustrated with numerical simulations. Then using a unified weighted energy estimates and technique of taking anti-derivatives, we prove that all types of traveling wave profiles, including non-monotone pulsating wave profiles, are nonlinearly and asymptotically stable if the initial data are small perturbations with zero mass from the spatially shifted traveling wave profiles.

AMS subject classifications: 35C07, 35K55, 35L65, 46N60, 92C17

Key words: Chemotaxis, conservation laws, traveling waves, nonlinear stability, weighted energy estimates.

1 Introduction

Chemotaxis, the movement of an organism or entity in response to a chemical stimulus, is a widespread phenomenon in nature. One of the pioneering chemotaxis models was proposed by Keller and Segel [17] as follows to describe the wave propagation of bacterial chemotaxis:

$$\begin{cases} u_t = du_{xx} - \chi[u(\ln w)_x]_x, \\ w_t = \varepsilon w_{xx} - \mu u w^m, \end{cases} \tag{1.1}$$

where u and w denote the cell density and chemical concentration, respectively. $d > 0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively, and $m \geq 0$ denotes

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the consumption rate. Generally chemotaxis is said to be attractive if the chemotactic coefficient $\chi > 0$ and repulsive if $\chi < 0$ with $|\chi|$ measuring the strength of chemotaxis. The chemotaxis model (1.1) was used in [17] to describe the process that bacteria u move up (i.e. $\chi > 0$) the concentration gradient of a nutrient (denoted by w) which is absorbed by the bacteria (i.e. $\mu > 0$). A prominent structural feature of (1.1) is that the chemotactic sensitivity function $\ln w$ is singular at $w=0$. This imposes tremendous challenges for both analysis and numerical computations. Though the existence/nonexistence of traveling wave solutions for any $m \geq 0$ have been well understood (cf. [29, 32]), the stability of traveling wave solutions was obtained only for the case $m=1$ and still widely remains open for the case $m \neq 1$. In the case $m=1$, if $\chi\mu > 0$, the following Cole-Hopf transformation (cf. [23, 34]):

$$v = \frac{\sqrt{\chi\mu}}{\mu} \frac{w_x}{w} \tag{1.2}$$

can remove the singularity and convert the system (1.1) into a non-singular system of conservation laws as follows:

$$\begin{cases} u_t + (uv)_x = du_{xx}, \\ v_t + (u + \sigma v^2)_x = \varepsilon v_{xx} \end{cases} \tag{1.3}$$

with

$$\sigma = -\frac{\varepsilon}{\chi},$$

where we have used the rescalings $\tilde{t} = \chi\mu t$ and $\tilde{x} = \sqrt{\chi\mu} x$ to rescale the model but suppressed the tildes for simplicity. The transformed system (1.3) has no singularity and is more tractable analytically. As $\chi > 0$ and hence $\sigma < 0$, a large amount of interesting results have been developed to the transformed system (1.3) for both $\varepsilon > 0$ and $\varepsilon = 0$, such as traveling wave solutions [1–3, 19–21, 23, 24], global well-posedness of large/small solutions in the whole space [6, 14, 30, 38] or in bounded domains (or intervals) with suitable boundary conditions [22, 31, 40] and boundary layer problem [15]. When $m \neq 1$, stability results are restricted only to the spectral stability [25] and absolute instability [5] for the case $m = \varepsilon = 0$ or instability [28] for $\varepsilon > 0, m = 0$. We also refer to a result in [8] for the existence of traveling wave solutions on a generalize Keller-Segel model. Results on the Keller-Segel model (1.1) with fractional diffusion are referred to [12, 13].

The afore-mentioned results are developed for the attractive case $\chi > 0$. For the repulsive case $\chi < 0$, the Keller-Segel model (1.1) with $m = 1$ and $\mu < 0$ was re-derived in [34] based on a random walk framework to describe the biased movement of cells that deposit signals modifying the local environment for subsequent movements, such as myxobacteria or ants that deposit non-diffusive or slowly-moving chemical substances on the way for succeeding passage. In this case, one can still use the Cole-Hopf transformation (1.2) to get (1.3) with $\sigma = -\varepsilon/\chi > 0$ for which there are also many mathematical results available for the global dynamics in bounded or unbounded domain (cf. [16, 37, 41, 42]). If $\sigma \geq 0$ is allowed to be an arbitrary constant, the system (1.3) may have more applications. For example, when $d=\varepsilon=0$ and $\sigma=1$, the system is the so-called Leroux's system describing fluid dynamics (see details in [10, 33]). When $d=\varepsilon=0$ and $\sigma=1/2$, the system (1.3) is called shallow water wave equation (cf. [9]).

As recalled above, when $\sigma=-\varepsilon/\chi<0$ and $\chi>0$, the system (1.3) has been extensively studied in the literature. In particular it was shown that the system (1.3) admits traveling wave solutions that are nonlinearly asymptotically stable if $\varepsilon>0$ is small or $\varepsilon=0$ (cf. [19] and references therein), where the wave profile for u is monotonically decreasing while the wave profile for v is monotonically increasing. However, the existence of traveling wave solutions of (1.3) with $\sigma>0$ has not been studied as far as we know. The purpose of this paper is to fill this gap by studying the existence and stability of traveling wave solutions of (1.3) with $\sigma>0$ in $\mathbb{R}$ subject to the following initial data:

$$(u,v)(x,0)=(u_0,v_0)(x) \longrightarrow (u_\pm,v_\pm) \quad \text{as} \quad x \to \pm\infty. \tag{1.4}$$

It turns out the existence and stability of traveling wave solutions of (1.3) with $\sigma>0$ have some substantial differences from the case $\sigma<0$. Firstly we find that the system (1.3) with $\sigma>0$ has three different types of traveling wave profiles (see Theorem 2.1) while (1.3) with $\sigma<0$ admits only one type of traveling wave profiles (see [19]). Secondly the stability of traveling wave solutions of (1.3) with $\sigma<0$ requires that $\varepsilon>0$ is small (see [19]), while in this paper we can establish the nonlinear stability of traveling wave solutions of (1.3) with $\sigma>0$ for any $\varepsilon>0$ (see Theorem 3.1).

The rest of this paper is organized as follows. In Section 2, we shall state the existence theorem of traveling wave solutions of (1.3) with $\sigma>0$ (see Theorem 2.1) and give details proofs based on phase plane analysis along with Poincaré-Bendixon theorem. Moreover, we use numerical simulations illustrate the traveling wave profiles. In Section 3, we shall state the stability theorem of traveling wave solutions (see Theorem 3.1) and prove it by a unified approach based on the weighted energy estimates and the technique of taking anti-derivative. Finally, we summarize our results and discuss the conversion of the results from the transformed system (1.3) to the original chemotaxis model (1.1).

2 Existence of traveling waves

In this section, we are devoted to studying the existence of traveling wave solutions.

2.1 Wave equations and critical points

A traveling wave solution of (1.3) in $(x,t)\in\mathbb{R}\times[0,\infty)$ is a non-constant solution in the form

$$(u,v)(x,t)=(U,V)(z), \quad z=x-st \tag{2.1}$$

with $U,V\in C^\infty(\mathbb{R})$ satisfying the boundary conditions

$$U(\pm\infty)=u_\pm, \quad V(\pm\infty)=v_\pm, \quad U_z(\pm\infty)=V_z(\pm\infty)=0, \tag{2.2}$$

where s is the wave speed assumed to be non-negative without loss of generality and z is called the wave variable. The constants $u_\pm$ and $v_\pm$ are called the asymptotic states of u and v, respectively, describing the asymptotic behavior of traveling wave profiles as $z\to\pm\infty$.

Substituting the wave ansatz (2.1) into PDE system (1.3) yields the traveling wave equations

$$\begin{cases} -sU_z+(UV)_z=dU_{zz}, \\ -sV_z+(U+\sigma V^2)_z=\varepsilon V_{zz}. \end{cases} \tag{2.3}$$

Integrating (2.3) with respect to z and using (2.2), we get

$$\begin{cases} dU_z=U(V-s)+c_1, \\ \varepsilon V_z=\sigma V^2-sV+U+c_2, \end{cases} \tag{2.4}$$

where

$$\begin{aligned} c_1&=-u_-(v_--s)=-u_+(v_+-s), \\ c_2&=-\sigma v_-^2+sv_--u_-=-\sigma v_+^2+sv_+-u_+ \end{aligned}$$

are constants. In what follows, we shall assume $c_1=c_2=0$ for simplicity. When c_1 or c_2 is non-zero, similar analysis and results can be carried out with more technicalities with the help of hyperbolic theory (cf. [23]). With $c_1=c_2=0$, ODE system (2.4) reduces to

$$\begin{cases} dU_z=U(V-s), & \text{(2.5a)} \\ \varepsilon V_z=\sigma V^2-sV+U. & \text{(2.5b)} \end{cases}$$

The nullclines of (2.5) satisfy

$$0=U(V-s), \quad U=-\sigma V^2+sV. \tag{2.6}$$

Solving (2.6), we obtain three critical points for the system (2.5) in the U-V plane as follows:

$$O=(0,0), \quad A=\left(0,\frac{s}{\sigma}\right), \quad B=\left((1-\sigma)s^2,s\right), \tag{2.7}$$

where the critical point A will coincide with B when $\sigma=1$.

Next we examine the local dynamics of (2.5) near each critical point. To this end, we write the Jacobian matrix of (2.5) at a critical point (u_*,v_*) as

$$J(u_*,v_*)=\begin{bmatrix} -\frac{s}{d}+\frac{v_*}{d} & \frac{u_*}{d} \\ \frac{1}{\varepsilon} & -\frac{s}{\varepsilon}+\frac{2\sigma}{\varepsilon}v_* \end{bmatrix}.$$

By the trace-determinant formula, we obtain the eigenvalue $\lambda_\pm$ of $J(u_*,v_*)$ as follows:

$$\lambda_\pm|_{(u_*,v_*)}=\frac{1}{2}\left\{\left(\frac{1}{d}+\frac{2\sigma}{\varepsilon}\right)v_*-s\left(\frac{1}{d}+\frac{1}{\varepsilon}\right)\pm\sqrt{\left(\frac{s}{d}-\frac{v_*}{d}-\frac{s}{\varepsilon}+\frac{2\sigma}{\varepsilon}v_*\right)^2+\frac{4}{d\varepsilon}u_*}\right\},$$

where $\lambda_- < \lambda_+$. Then we have

$$\begin{aligned}(\lambda_-+\lambda_+)|_{(u_*,v_*)} &= \left(\frac{1}{d}+\frac{2\sigma}{\varepsilon}\right)v_* - s\left(\frac{1}{d}+\frac{1}{\varepsilon}\right),\\ (\lambda_-\lambda_+)|_{(u_*,v_*)} &= \left(\frac{v_*}{d}-\frac{s}{d}\right)\left(\frac{2\sigma}{\varepsilon}v_*-\frac{s}{\varepsilon}\right)-\frac{u_*}{d\varepsilon}.\end{aligned} \tag{2.8}$$

When $s=0$, (2.5) has only one trivial critical point $(0,0)$ and the eigenvalues of the Jacobian matrix at $(0,0)$ are $\lambda_\pm = 0$. Therefore, $(0,0)$ is a center and there will be no traveling waves in this case. In what follows, we proceed with the case $s>0$ and find the local dynamics of each critical point as follows:

- At the critical point $O=(0,0)$, we have
$$(\lambda_-+\lambda_+)|_{(0,0)} = -s\left(\frac{1}{d}+\frac{1}{\varepsilon}\right)<0, \quad (\lambda_-\lambda_+)|_{(0,0)} = \frac{s^2}{d\varepsilon}>0.$$
Therefore, $\lambda_- < \lambda_+ < 0$ and O is a stable node.
- At the critical point $A=(0,s/\sigma)$, we can calculate that
$$\lambda_-+\lambda_+ = \frac{s}{d}\left(\frac{1}{\sigma}-1\right)+\frac{s}{\varepsilon}, \quad \lambda_-\lambda_+ = \frac{s^2}{d\varepsilon}\left(\frac{1}{\sigma}-1\right).$$
Therefore, we have the following cases. When $0<\sigma<1$, then $\lambda_-\lambda_+>0$ and $\lambda_-+\lambda_+>0$ and hence A is an unstable node; when $\sigma=1$, then $\lambda_-=0$ and $\lambda_+=s/\varepsilon>0$ and the critical point A is unstable with one unstable manifold and one central manifold; when $\sigma>1$, it has that $\lambda_-\lambda_+<0$ and hence A is a saddle.
- At the critical point $B=((1-\sigma)s^2,s)$, one has
$$\lambda_-+\lambda_+ = \frac{s}{\varepsilon}(2\sigma-1), \quad \lambda_-\lambda_+ = \frac{\sigma-1}{d\varepsilon}s^2.$$
We also have three cases to discuss. If $0<\sigma<1$, then $\lambda_-\lambda_+<0$ and hence B is a saddle; if $\sigma>1$, then $\lambda_-+\lambda_+>0$ and $\lambda_-\lambda_+>0$, and hence B is an unstable node; if $\sigma=1$, then $\lambda_-=0$ and $\lambda_+=s/\varepsilon>0$. In this case, the critical point B is unstable with one unstable manifold and one central manifold.

We recap the aforementioned results in the following lemma.

Lemma 2.1. *Considering the critical points of* (2.5), *we have the following conclusions:*

(i) The critical point $O=(0,0)$ is a stable node for any $d,\varepsilon,\sigma>0$.

(ii) The critical points $A=(0,s/\sigma)$ and $B=((1-\sigma)s^2,s)$ have the properties as follows:

- *If $0<\sigma<1$, then $A=(0,s/\sigma)$ is an unstable node and $B=((1-\sigma)s^2,s)$ is a saddle.*
- *If $\sigma=1$, then $A=B=(0,s/\sigma)$ and they are linearly unstable with one unstable manifold and one central manifold.*
- *If $\sigma>1$, then $A=(0,s/\sigma)$ is a saddle and $B=((1-\sigma)s^2,s)$ is an unstable node.*

2.2 Existence and proofs

Based on the results provided in Lemma 2.1, below we show the existence of traveling wave solutions of (1.3) by phase plane analysis. Specifically, we prove the following theorem.

Theorem 2.1. *Let d,ε and σ be positive constants and consider the problem* (1.3)-(1.4) *in* $\mathbb{R}$ *with* $u_\pm \geq 0$ *and* $v_\pm \geq 0$. *Then the following results hold:*

(i) If $\sigma \geq 1$, the system (1.3)-(1.4) *does not have non-negative traveling wave solutions.*

(ii) If $0<\sigma<1$, we have the following conclusions for given constant $v_- > 0$:

(a) If $v_+ = 0$ and $u_- > 0$, then there is a unique wave speed $s = v_-$ such that the system (1.3)-(1.4) *has a traveling wave solution (U,V) satisfying* (2.5) *and* (2.2) *with $u_- = (1-\sigma)v_-^2$, $u_+ = 0$ and $U_z < 0, V_z < 0$, which is unique up to a translation.*

(b) If $v_+ > 0$, then there is a unique wave speed $s = \sigma v_-$ such that the system (1.3)-(1.4) *has a traveling wave solution (U,V) satisfying* (2.5) *and* (2.2) *with $u_- = 0, u_+ = (1-\sigma)s^2$ and $v_+ = \sigma v_-$ and $U_z > 0, V_z < 0$, which is unique up to a translation.*

(c) If $v_+ = 0$ and $u_- = 0$, then there is a unique wave speed $s = \sigma v_-$ such that the system (1.3)-(1.4) *has a traveling wave solution (U,V) satisfying* (2.5) *and* (2.2) *with $u_+ = 0$ and $V_z < 0$, where the profile U is non-monotone and there is a point $z_0 \in \mathbb{R}$ such that $U_z > 0$ for $z \in (-\infty, z_0)$ and $U_z < 0$ for $z \in (z_0, \infty)$.*

Remark 2.1. It was known (cf. [19]) that the system (1.3) with $\sigma<0$ admits only one type of traveling wave profile (U,V) with $U_z<0$ and $V_z>0$. However, the results of Theorem 2.1 assert that the system (1.3) with $\sigma>0$ may admit three different types of traveling wave profiles and none of them is similar to the case $\sigma<0$. In particular the non-monotone pulsating wave profile for U is possible (see Fig. 5 for illustration), which is substantially different from the case $\sigma<0$. This also indicates that attractive and repulsive chemotaxis may result in very different population spreading dynamics. From mathematical point of view, the sign of σ is of importance to determine the dynamics of the conservational system (1.3).

2.3 Case of $\sigma \geq 1$

In the case $\sigma = 1$, $A = B = (0,s)$ and the system (2.5) admits only two critical points $O(0,0)$ and $A(0,s)$, where $O(0,0)$ is a stable node (see Lemma 2.1). The Jacobian matrix J at $A(0,s)$ is

$$J|_A = \begin{bmatrix} 0 & 0 \\ 1/\varepsilon & s/\varepsilon \end{bmatrix}.$$

The eigenvalues of $J|_A$ are $\lambda_- = 0, \lambda_+ = s/\varepsilon > 0$ and the eigenvector associated with λ_+ is $(0,\alpha)$ with $\alpha \in \mathbb{R}\backslash\{0\}$. Since the V-axis (i.e. $U=0$) is an invariant set of (2.5), the unstable manifold emanating from A will escape along the V-axis towards $O(0,0)$. Hence, there are no non-trivial orbits connecting A and O to generate a traveling wave solution.

We proceed to consider the case $\sigma > 1$. In this case, the critical point $B = ((1-\sigma)s^2,\sigma)$ has a negative component. Since the negative solution loses its biological relevance, we ignore the critical point B in this case and turn to consider the possible heteroclinic orbit connecting A to O. Similar to the above analysis, we can find the eigenvalues of the Jacobian matrix J at A are $\lambda_+ = s/\varepsilon > 0, \lambda_- = (s/d)(1/\sigma - 1) < 0$, which entails that A is a saddle. Since O is a stable node, we need to examine the eigenvector associated with the eigenvalue λ_+, which is $(0,\alpha)$ for any constant $\alpha > 0$. We know that the set $\{(U,V):U=0\}$ is invariant. Thus, the unstable manifold emanating from the critical point A will proceed along the V-axis and hence there are no non-trivial orbits connecting A and O, similar to the case $\sigma = 1$. Consequently there are no traveling wave solutions in this case. This proves Theorem 2.1(i).

2.4 Case of $0 < \sigma < 1$

In this case, we know from Lemma 2.1 that $O(0,0)$ is a stable node, $A = (0,s/\sigma)$ is an unstable node and $B = ((1-\sigma)s^2,s)$ is a saddle. Hence, in principle there exist three possible heteroclinic orbits connecting $B = ((1-\sigma)s^2,s)$ to $O = (0,0)$, $A = (0,s/\sigma)$ to $B = ((1-\sigma)s^2,s)$ and $A = (0,s/\sigma)$ to $O = (0,0)$. Each of them will generate a traveling wave solution to (1.3). We use the "pplane" program in Matlab to plot the phase portrait of (2.5) with $s = 1$ in Fig. 1, where these three possible heteroclinic orbits are all observed. Next we shall use phase plane analysis to prove the existence of these heteroclinic orbits which yield three different traveling wave profiles of (1.3) recorded in Theorem 2.1(ii).

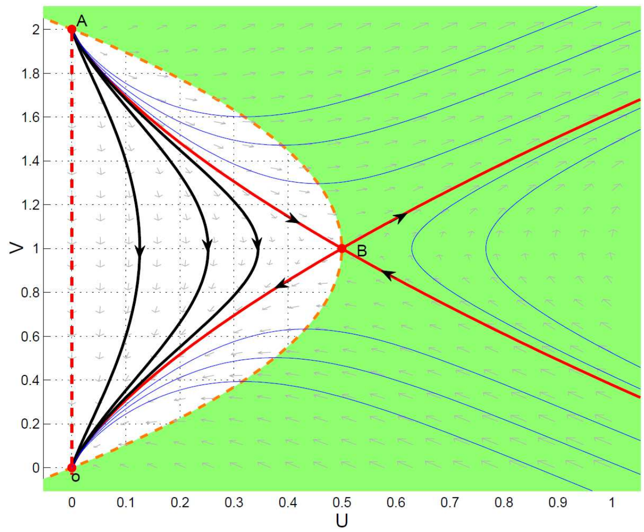

Figure 1: A phase portrait of (2.5) with $\sigma = 0.5$ and $s = 1$, where the region enclosed by the dashed curves (unshaded region) is the region relevant to generate traveling wave solutions.

2.4.1 Heteroclinic orbit connecting $B=((1-\sigma)s^2,s)$ to $O=(0,0)$

From Lemma 2.1, we know that $O=(0,0)$ is a stable node and $B=((1-\sigma)s^2,s)$ is a saddle. Hence, there might be a heteroclinic orbit connecting the critical point $B=((1-\sigma)s^2,s)$ to $O=(0,0)$. Next we shall verify this possibility. To this end, we construct a simply connected open set R_1 as follows:

$$R_1=\{(U,V):0<U\leq -\sigma V^2+sV, 0<V<s\},$$

and show that all the orbits within R_1 can not leave the region R_1. Clearly, the region R_1 is bounded by the following curves (see a schematic diagram in Fig. 2(a)):

$$\begin{aligned}
&\Gamma_1=\{(U,V)\,|\,U=-\sigma V^2+sV, 0<V<s\},\\
&\Gamma_2=\{(U,V)\,|\,0\leq U\leq (1-\sigma)s^2, V=s\},\\
&\Gamma_3=\{(U,V)\,|\,U=0, 0<V<s\}.
\end{aligned}$$

Along Γ_1, we have

$$V_z=\frac{1}{\varepsilon}(\sigma V^2-sV+U)=0,\quad U_z=\frac{U}{d}(V-s)<0.$$

Hence, the vector filed of (2.5) will cross the curve Γ_1 and point leftward to the inside of R_1. Along Γ_2, we have

$$V_z=\frac{1}{\varepsilon}(U+\sigma V^2-sV)<0,\quad U_z=0,$$

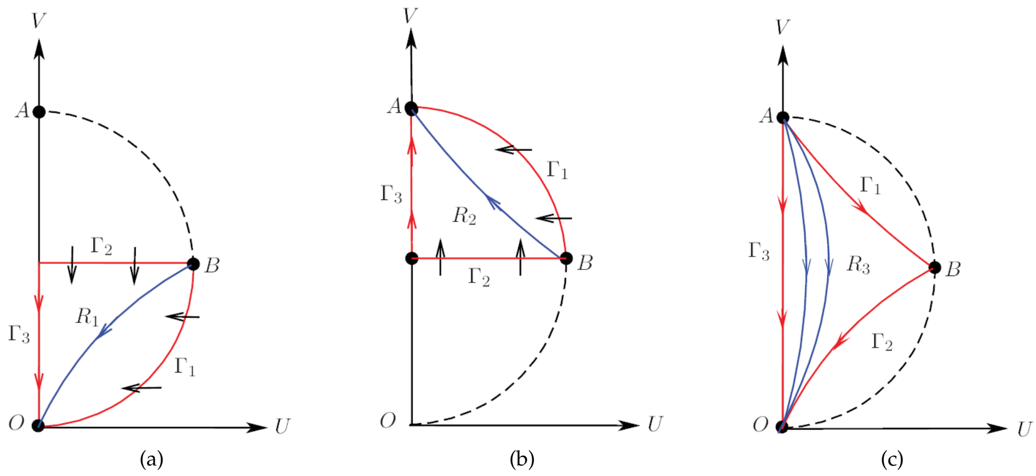

Figure 2: Schematic invariant regions constructed for the system (2.5) (see (a) and (c)) and (2.10) (see (b)), where we choose $s=1$ and $\sigma=0.5$.

and thus the vector field of (2.5) will pass through Γ_2 and point downward. Finally, along Γ_3,

$$V_z=\frac{1}{\varepsilon}(\sigma V^2-sV)=\frac{V}{\varepsilon}(\sigma V-s)<0$$

since $0<\sigma<1$ and $V<s$, and $U_z=U(V-s)/d=0$, which imply that the orbits of (2.5) can only move along Γ_3 towards O. Hence, we show that all orbits inside R_1 can not pass through the boundary and leave R_1. We proceed to show that the unstable manifold emanating from the saddle point B must enter the region R_1. It is easy to calculate that

$$\left.\frac{dV}{dU}\right|_B^{\Gamma_1}=\frac{1}{s(1-2\sigma)},\quad \left.\frac{dV}{dU}\right|_B^{\Gamma_2}=0,$$

where $(dV/dU)|_B^{\Gamma_i}$ denotes the slope of the tangent line of Γ_i at B. Now we examine the direction of the unstable manifold of the critical point B, which is tangential to the eigenvector of the positive eigenvalue of the Jacobian matrix at B

$$J|_B=\begin{bmatrix} 0 & \frac{s^2}{d}(1-\sigma) \\ \frac{1}{\varepsilon} & \frac{s}{\varepsilon}(2\sigma-1)\end{bmatrix}.$$

Clearly, the eigenvalues of $J|_B$ are

$$\lambda_\pm=\frac{s}{2\varepsilon}\left[2\sigma-1\pm\sqrt{(2\sigma-1)^2+\frac{4\varepsilon}{d}(1-\sigma)}\right]$$

with $\lambda_-<0$ and $\lambda_+>0$. The eigenvector associated with the positive eigenvalue λ_+ is $\vec{e}=(\varepsilon\lambda_++s(1-2\sigma),1)^\top$. Hence, the slope of the unstable manifold of (2.5) at the critical point B is

$$\left.\frac{dV}{dU}\right|_B=\frac{1}{\varepsilon\lambda_++s(1-2\sigma)}.$$

If $\sigma\le 1/2$, then it follows that

$$\left.\frac{dV}{dU}\right|_B^{\Gamma_1}>\left.\frac{dV}{dU}\right|_B>\left.\frac{dV}{dU}\right|_B^{\Gamma_2}=0$$

since $\lambda_+>0$ (see an illustration in Fig. 2(a) for $\sigma=1/2$). If $\sigma>1/2$, then $(dV/dU)|_B^{\Gamma_1}<0$ and

$$\left.\frac{dV}{dU}\right|_B^{\Gamma_2}=0<\left.\frac{dV}{dU}\right|_B\le\infty,\quad \text{if}\quad \sigma>\frac{1}{2}\quad\text{and}\quad \varepsilon\lambda_++s(1-2\sigma)\ge 0$$

or

$$\left.\frac{dV}{dU}\right|_B<\left.\frac{dV}{dU}\right|_B^{\Gamma_1}<\left.\frac{dV}{dU}\right|_B^{\Gamma_2}=0,\quad \text{if}\quad \sigma>\frac{1}{2}\quad\text{and}\quad \varepsilon\lambda_++s(1-2\sigma)<0.$$

The above inequalities indicate that angle between the eigenvector $\vec{e}$ and the horizontal line Γ_2 is less then the angle between the tangent line of Γ_1 at B and the horizontal line Γ_2. Hence, the unstable manifold of (2.5) emanating from the saddle point B must enter the region R_1. We next show this unstable manifold (orbit) will converge to the critical point O by the Poincaré-Bendixson theorem. To this end, we need to prove there are no closed (or periodic) orbits lying within R_1, which can be verified by the Bendixson-Dulac theorem. Indeed, we define a Dulac function $h(U,V)=U^{-\gamma}$ with $\gamma>0$ and rewrite (2.5) as

$$U_z=f(U,V), \quad V_z=g(U,V), \tag{2.9}$$

where

$$f(U,V)=\frac{1}{d}U(V-s), \quad g(U,V)=\frac{1}{\varepsilon}(\sigma V^2-sV+U).$$

Then one can directly compute that

$$\begin{aligned}\Lambda(U,V):&=\frac{\partial}{\partial U}(hf)+\frac{\partial}{\partial V}(hg)=h\left(\frac{\partial f}{\partial U}+\frac{\partial g}{\partial V}\right)+f\frac{\partial h}{\partial U}\\&=U^{-\gamma}\left[\left(\frac{1-\gamma}{d}+\frac{2\sigma}{\varepsilon}\right)(V-s)+\frac{(2\sigma-1)s}{\varepsilon}\right].\end{aligned}$$

We shall show that $\Lambda(U,V)\neq 0$ within the region R_1 by choosing appropriate $\gamma>0$. First note $V-s<0$ in R_1. If $0<\sigma\leq 1/2$, it is clear that $\Lambda(U,V)<0$ if we choose $0<\gamma<1$. While if $\sigma>1/2$, we may choose $\gamma>1$ suitably large so that $(1-\gamma)/d+2\sigma/\varepsilon<0$ and hence $\Lambda(U,V)>0$. Hence, we are able to choose suitable $\gamma>0$ so that

$$\frac{\partial}{\partial U}(hf)+\frac{\partial}{\partial V}(hg)\neq 0$$

within the region R_1. Since the region R_1 is simply connected and $h(U,V)$ is C^1, by the Bendixson-Dulac theorem (cf. [18,39]), the dynamical system (2.5) has no periodic orbits inside the region R_1. Further by the Poincaré-Bendixson theorem, we conclude that the orbit (i.e. unstable manifold) emanating from the saddle point B must converge to O as $z\to\infty$. This orbit generates a traveling wave solution (U,V) connecting $B=((1-\sigma)s^2,s)$ to $O=(0,0)$. Since the traveling wave ODE system (2.5) is autonomous, if $(U,V)(z)$ is a solution, then so is $(U,V)(z)=(U,V)(z-z_0)$ for any constant z_0, which has the same orbit as $(U,V)(z)$ and corresponds to a traveling wave solution of the same speed that is translated by a constant distance z_0. Since B is a saddle point, there is only one unstable manifold emanating from B and entering the region R_1 and hence this heteroclinic orbit is unique up to a translation.

Since $V<s$ and $\sigma V^2-sV+U<0$ inside the region R_1, then it follows that

$$U_z=\frac{1}{d}U(V-s)<0, \quad V_z=\frac{1}{\varepsilon}(\sigma V^2-sV+U)<0,$$

which implies that the traveling wave profiles U and V obtained above are monotonically decreasing. This completed the proof of Theorem 2.1(ii)-(a). We use Matlab PDEPE solver

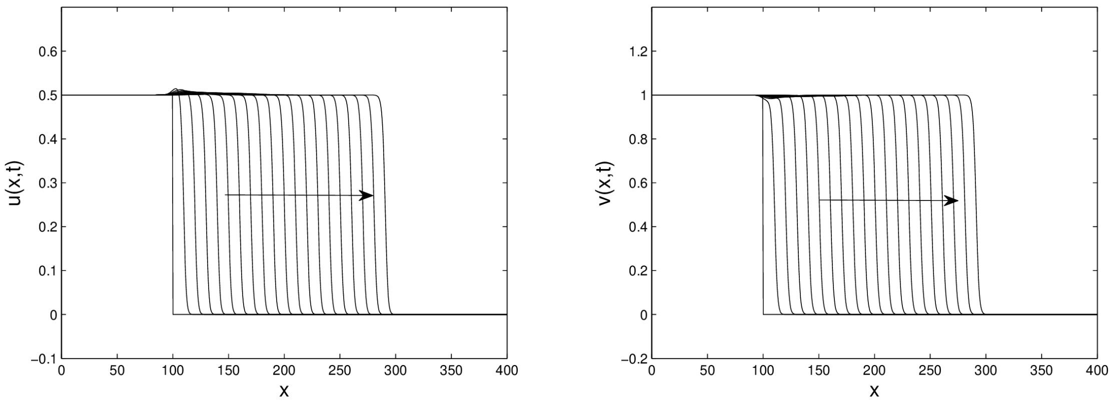

Figure 3: Numerical simulations of traveling waves generated by the model (1.3) connecting B to O, where $d=\varepsilon=1$ and $\sigma=0.5$. The arrows indicates the wave propagation direction as time evolves and each curve represents the solution profile at a certain time spaced by $t=10$ starting from the initial value (u_0,v_0) plotted at the far left.

to numerically solve the system (1.3) and the numerical simulations of solutions are plotted in Fig. 3 where we do obverse monotonically decreasing traveling waves propagating as time evolves.

2.4.2 Heteroclinic orbit connecting $A=(0,s/\sigma)$ to $B=((1-\sigma)s^2,s)$

From Lemma 2.1, we know that $A=(0,s/\sigma)$ is an unstable node and $B=((1-\sigma)s^2,s)$ is a saddle. Hence, there might be a heteroclinic orbit connecting $A=(0,s/\sigma)$ to $B=((1-\sigma)s^2,s)$. Since $A=(0,s/\sigma)$ is an unstable node, it is impossible to construct an invariant set containing A as a boundary or interior point directly. Below we develop an idea by reversing the direction of the possible orbit connecting $A=(0,s/\sigma)$ and $B=((1-\sigma)s^2,s)$. That is, we set $\xi=-z$ and rewrite the ODE system (2.5) as

$$\begin{cases} dU_\xi=-U(V-s), \\ \varepsilon V_\xi=-\sigma V^2+sV-U. \end{cases} \tag{2.10}$$

Now $A=(0,s/\sigma)$ is a stable node for the new system (2.10) with independent variable ξ while $B=((1-\sigma)s^2,s)$ is still a saddle. If we can show that (2.10) has a heteroclinic orbit connecting $B=((1-\sigma)s^2,s)$ to $A=(0,s/\sigma)$, then the same heteroclinic orbit with reversed direction gives a heteroclinic orbit connecting $A=(0,s/\sigma)$ to $B=((1-\sigma)s^2,s)$ for the system (2.5). Next we focus on the proof of the existence of heteroclinic orbit connecting $B=((1-\sigma)s^2,s)$ to $A=(0,s/\sigma)$ for the ODE system (2.10). To this end, we construct an open and simply connected region R_2 bounded by the following curves (see Fig. 2(b) for an illustration):

$$\begin{aligned} &\Gamma_1=\{(U,V)\,|\,U=-\sigma V^2+sV, s<V<s/\sigma\}, \\ &\Gamma_2=\{(U,V)\,|\,0\le U\le (1-\sigma)s^2, V=s\}, \\ &\Gamma_3=\{(U,V)\,|\,U=0, s<V<s/\sigma\}. \end{aligned}$$

Applying the same arguments as in Section 2.4.1, we can show the ODE system (2.10) admits a heteroclinic orbit connecting $B=((1-\sigma)s^2,s)$ to $A=(0,s/\sigma)$ which is unique up to a translation. For brevity, we omit the details here. Reversing the direction of this orbit, we get a heteroclinic orbit connecting $A=(0,s/\sigma)$ to $B=((1-\sigma)s^2,s)$ for the original ODE system (2.5), which yields a traveling wave solution (U,V) to (1.3) unique up to a translation. Since $U<-\sigma V^2+sV$ and $s<V<s/\sigma$ within R_2, we have $U_\xi=-U(V-s)/d<0$ and $V_\xi=(-\sigma V^2+sV-U)/\varepsilon>0$. Hence, $U_z=-U_\xi>0$ and $V_z=-V_\xi<0$. This completes the proof of Theorem 2.1(ii)-(b). We use numerical simulation to illustrate the traveling wave profiles shown in Fig. 4 for this case, which are well consistent with the qualitative results stated in Theorem 2.1(ii)-(b).

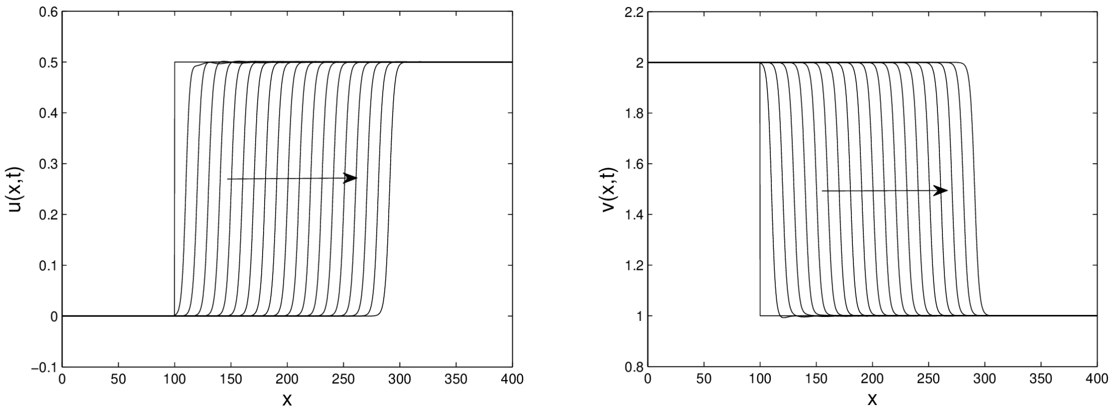

Figure 4: Numerical simulations of traveling waves generated by the model (1.3) connecting A to B, where $d=\varepsilon=1$ and $\sigma=0.5$. The arrows indicates the wave propagation direction as time evolves and each curve represents the solution profile at a certain time spaced by $t=10$ starting from the initial value (u_0,v_0) plotted at the far left.

2.4.3 Heteroclinic orbit connecting $A(0,s/\sigma)$ to $O(0,0)$

Now we prove that the system (2.5) admits heteroclinic orbits connecting $A(0,s/\sigma)$ to $O(0,0)$. In Section 2.4.2, we show that there is a heteroclinic orbit connecting $A(0,s/\sigma)$ to $B((1-\sigma)s^2,s)$, which indeed is a separatrix denoted by Γ_1 in the following. In Section 2.4.1, it is shown that there is a heteroclinic orbit connecting $B((1-\sigma)s^2,s)$ to $O(0,0)$, which is another separatrix denoted by Γ_2. To prove there is a heteroclinic orbit connecting $A(0,s/\sigma)$ to $O(0,0)$, we now consider a simply connected open region R_3 bounded by Γ_1,Γ_2 and Γ_3, where Γ_3 denotes the segment OA, as shown in Fig. 2(c). We have previously shown that A is an unstable node and O is a stable node. Hence, any unstable manifold emanating from A and pointing into the region R_3 can not touch or intersect curves Γ_i $(i=1,2,3)$ because of the uniqueness of solutions to the ODE system (2.5). Next, we shall apply the Bendixson-Dulac theorem to show that there are no closed/periodic orbits existing within the region R_3. For this purpose, we construct a Dulac function: $\varphi(U,V)=V^{-\theta}$, where $\theta\gg1$. By (2.9), one can directly compute that

$$
\begin{aligned}
&\frac{\partial}{\partial U}(\varphi f)+\frac{\partial}{\partial V}(\varphi g)\\
&=\left(\frac{1}{d}+\frac{2\sigma}{\varepsilon}-\frac{\theta\sigma}{\varepsilon}\right)V^{-\theta+1}+sV^{-\theta}\left(\frac{\theta}{\varepsilon}-\frac{1}{d}-\frac{1}{\varepsilon}\right)-\frac{\theta}{\varepsilon}UV^{-\theta-1}\\
&=V^{-\theta-1}\left(\frac{1}{d}+\frac{(2-\theta)\sigma}{\varepsilon}\right)\left(V^2+\frac{d\theta-\varepsilon-d}{\varepsilon+2d\sigma-d\sigma\theta}sV-\frac{\theta U}{\varepsilon/d+2\sigma-\sigma\theta}\right).
\end{aligned}
$$

It is straightforward to verify that

$$
\lim_{\theta\to\infty}\left(V^2+\frac{d\theta-\varepsilon-d}{\varepsilon+2d\sigma-d\sigma\theta}sV-\frac{\theta U}{\varepsilon/d+2\sigma-\sigma\theta}\right)=\frac{1}{\sigma}(\sigma V^2-sV+U).
$$

Since the region R_3 is always on the left to the parabola $U=-\sigma V^2+sV$ (see Fig. 2(c)), we have $\sigma V^2-sV+U<0$ in R_3. It is also clear that $1/d+(2-\theta)\sigma/\varepsilon<0$ as long as $\theta>\varepsilon/(d\sigma)+2$. Therefore, by choosing $\theta>0$ sufficiently large, we can have

$$
\frac{\partial}{\partial U}(\varphi f)+\frac{\partial}{\partial V}(\varphi g)>0
$$

in R_3. This means that $(\partial/\partial U)(\varphi f)+(\partial/\partial V)(\varphi g)\neq 0$ everywhere in the region R_3. Hence, the Bendixson-Dulac theorem entails that no periodic orbit is contained within the region R_3. From the Poincaré-Bendixson theorem, the orbit emanating from the unstable manifolds of A and pointing into the region R_3 has to converge to the critical points O or B. But it can not converge to the critical point B since it is a saddle point and Γ_1 is a separatrix. Therefore, this orbit must converge to the stable node O. Since there are infinite many outgoing unstable manifolds from A entering the region R_3, there will be infinite many such heteroclinic orbits connecting A to O, which generate infinite many traveling wave solutions. Since $U<-\sigma V^2+sV$ in the region R_3, it follows that

$$
V_z=\frac{1}{\varepsilon}(\sigma V^2-sV+U)<0.
$$

Due to the facts that $V\in(0,s/\sigma)$ and $0<\sigma<1$, there is a unique $z_0\in\mathbb{R}$ such that $V(z_0)=s$, which implies $V>s$ if $z<z_0$ while $V<s$ if $z>z_0$. Noticing that $U_z=U(V-s)/d$, we have $U_z>0$ if $z<z_0$ and $U_z<0$ if $z>z_0$. Collecting the above results for V, one finds U_z will change the sign once at $z=z_0$, namely the traveling wave profile U will change the monotonicity once. This completes the proof of Theorem 2.1(ii)-(c). We plot the numerical simulations of such traveling wave profiles in Fig. 5, which well agree with the results recorded in Theorem 2.1(ii)-(c).

3 Nonlinear asymptotic stability

In this section, we prove the nonlinear asymptotic stability of the traveling wave solutions (U,V) obtained in Theorem 2.1. Precisely, we show that the solution of (1.3)-(1.4)

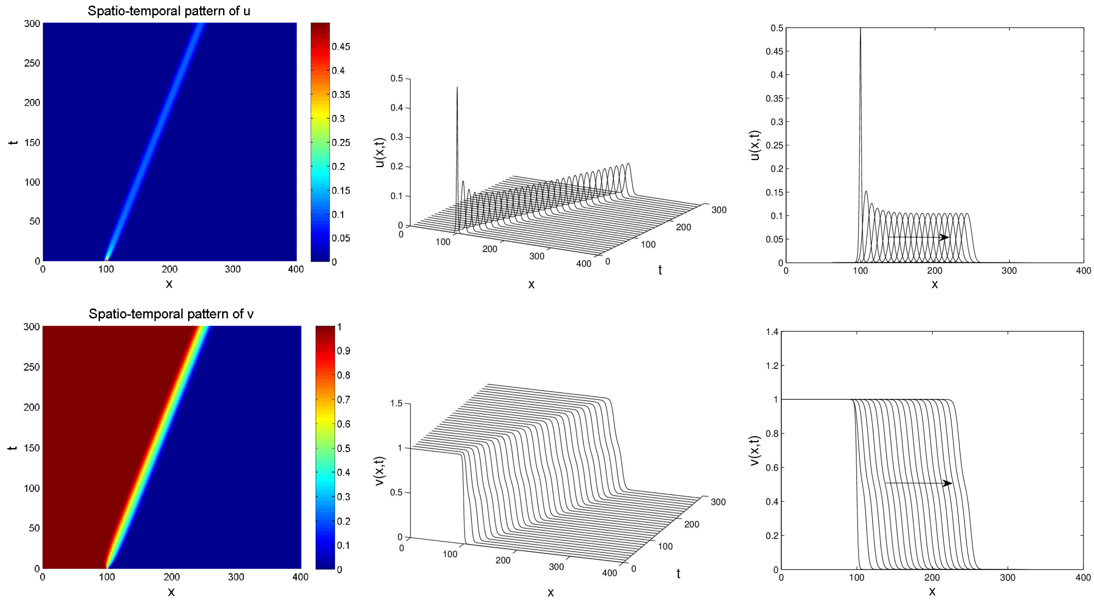

Figure 5: Numerical simulations of traveling waves generated by the model (1.3) connecting A to O plotted in different ways, where $\varepsilon=d=1$ and $\sigma=0.5$ and initial value (u_0,v_0) is given by $u_0=\exp(x-100)/(1+\exp(2(x-100))), v_0=1/(1+\exp((x-100)))$.

approaches to a traveling wave solution $(U,V)(x-st)$, properly translated by an amount x_0, i.e.

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(U,V)(x+x_0-st)| \to 0 \quad \text{as} \quad t \to +\infty,$$

where x_0 satisfies the following identity derived from the principle of conservation of mass (cf. [35]):

$$\int_{-\infty}^{+\infty}\begin{pmatrix} u_0(x)-U(x) \\ v_0(x)-V(x)\end{pmatrix}dx = x_0\begin{pmatrix} u_+-u_- \\ v_+-v_-\end{pmatrix}+\beta r_1(u_-,v_-),$$

where $r_1(u_-,v_-)$ denotes the first right eigenvector of the Jacobian matrix of (1.3) with $d=\varepsilon=0$ at (u_-,v_-). The coefficient β yields the diffusion wave in general [35]. Both β and x_0 are uniquely determined by the initial data (u_0,v_0). For the stability of small-amplitude shock waves of conservation laws with diffusion waves (i.e. $\beta\neq 0$), we refer to [27,36] for details. In the present paper, we will neglect the diffusion wave by assuming $\beta=0$ but consider the stability of large-amplitude waves (i.e. $|u_--u_+|$ and $|v_--v_+|$ can be arbitrarily large). The stability of large-amplitude traveling waves of conservation laws is a prominent question, and there are no results for general conservation laws (cf. [11,26,27]) except some special systems (cf. [19,23]). Then by the conserved equations in (1.3), we can show that

$$\int_{\mathbb{R}}\begin{pmatrix} u(x,t)-U(x+x_0-st) \\ v(x,t)-V(x+x_0-st)\end{pmatrix}dx = \int_{\mathbb{R}}\begin{pmatrix} u_0(x)-U(x+x_0) \\ v_0(x)-V(x+x_0)\end{pmatrix}dx$$

$$
\begin{aligned}
&=\int_{\mathbb{R}}\begin{pmatrix}u_0(x)-U(x)\\ v_0(x)-V(x)\end{pmatrix}dx+\int_{\mathbb{R}}\begin{pmatrix}U(x)-U(x+x_0)\\ V(x)-V(x+x_0)\end{pmatrix}dx\\
&=\int_{\mathbb{R}}\begin{pmatrix}u_0(x)-U(x)\\ v_0(x)-V(x)\end{pmatrix}dx-x_0\begin{pmatrix}u_+-u_-\\ v_+-v_-\end{pmatrix}=\vec{0}. \qquad (3.1)
\end{aligned}
$$

This allows us to employ the technique of taking anti-derivative to decompose the solution as

$$(u,v)(x,t)=(U,V)(x+x_0-st)+(\phi_z,\psi_z)(z,t), \tag{3.2}$$

where $z=x-st$. That is

$$\big(\phi(z,t),\psi(z,t)\big)=\int_{-\infty}^{z}\big(u(y,t)-U(y+x_0-st),v(y,t)-V(y+x_0-st)\big)dy$$

for all $z\in\mathbb{R}$ and $t\geq 0$. It then follows from (3.1) that

$$\phi(\pm\infty,t)=\psi(\pm\infty,t)=0,\quad \forall t\geq 0.$$

The initial data of (ϕ,ψ) is thus given by

$$(\phi_0,\psi_0)(z)=\int_{-\infty}^{z}\big(u_0(y)-U(y+x_0),v_0(y)-V(y+x_0)\big)dy \tag{3.3}$$

with $(\phi_0,\psi_0)(\pm\infty)=0$.

Before stating our results on the stability of traveling wave solutions, we shall introduce some notation. In the sequel, $\int_{-\infty}^{\infty}f(x,t)dx$ and $\int_0^t\int_{-\infty}^{\infty}f(x,t)dxdt$ will be abbreviated as $\int f(x,t)$ and $\int_0^t\int f(x,t)$. $H_w^k(\mathbb{R})$ denotes the space of measurable functions f so that $\sqrt{w}\partial_x^j f\in L^2(\mathbb{R})$ for $0\leq j\leq k$ with norm

$$\|f\|_{H_w^k(\mathbb{R})}:=\left(\sum_{j=0}^{k}\int w(x)|\partial_x^j f|^2dx\right)^{\frac{1}{2}}.$$

For simplicity, the conventions

$$\|f\|:=\|f\|_{L^2(\mathbb{R})},\quad \|f\|_w:=\|\sqrt{w}f\|_{L^2(\mathbb{R})},\quad \|f\|_k:=\|f\|_{H^k(\mathbb{R})},\quad \|f\|_{k,w}:=\|\cdot\|_{H_w^k(\mathbb{R})}$$

for $k=1,2,\ldots$ will be used. We also use C to denote a generic positive constant independent of time t which may vary in the context.

Theorem 3.1. *Let $(U,V)(x-st)$ be a traveling wave solution obtained in Theorem* 2.1. *Assume that there exists a constant x_0 such that the initial perturbation from the spatially shifted traveling wave profile (U,V) with shift x_0 is of integral zero, namely the initial value (ϕ_0,ψ_0) defined in* (3.3) *satisfies $(\phi_0,\psi_0)(\pm\infty)=0$. Then there exists a constant $\delta>0$ such that if*

$$\|u_0-U\|_{1,w}+\|v_0-V\|_1+\|\phi_0\|_w+\|\psi_0\|\leq\delta,$$

the Cauchy problem (1.3)-(1.4) *has a unique global solution* $(u,v)(x,t)$ *satisfying*

$$(u-U,v-V)\in C\big([0,\infty);H_w^1\times H^1\big)\cap L^2\big((0,\infty);H_w^2\times H^2\big),$$

where the weight function $w(z):=1/U(z)$ *for* $z\in\mathbb{R}$*. Furthermore, the solution has the following asymptotic stability:*

$$\sup_{x\in\mathbb{R}}|(u,v)(x,t)-(U,V)(x+x_0-st)| \;\to\; 0 \quad \text{as} \quad t \;\to\; +\infty.$$

Remark 3.1. When $\sigma<0$, similar results on the nonlinear asymptotic stability of traveling wave solutions have been obtained in [19] for $\varepsilon>0$ small. But the nonlinear asymptotic stability results in Theorem 3.1 for $\sigma>0$ hold true for any $\varepsilon>0$. This is another essential difference between $\sigma<0$ and $\sigma>0$, in addition to the differences on the traveling wave profiles discussed in Remark 2.1. Moveover, it is worthwhile to note that the nonlinear asymptotic stability in Theorem 3.1 particularly holds true for the case that U is a non-monotone pulsating wave profile since the unified approach used to prove the stability result does not depend on the monotonicity of U.

To show Theorem 3.1, we first derive the equations for (ϕ,ψ). Indeed, substituting (3.2) into (1.3), assuming $x_0=0$ without loss of generality, using (2.3) and integrating the resulting system in z, after some calculations, we get

$$\begin{cases}\phi_t=d\phi_{zz}+(s-V)\phi_z-U\psi_z-\phi_z\psi_z, & z\in\mathbb{R},\quad t>0, \qquad (3.4a)\\ \psi_t=\varepsilon\psi_{zz}+(s-2\sigma V)\psi_z-\phi_z-\sigma\psi_z^2, & z\in\mathbb{R},\quad t>0, \qquad (3.4b)\\ \phi(\pm\infty,t)=\psi(\pm\infty,t)=0. & \qquad (3.4c)\end{cases}$$

We look for solutions of system (3.4) in the following solution space:

$$\begin{aligned}X(0,T):=\big\{\big(\phi(z,t),\psi(z,t)\big):&\phi\in C\big([0,T];H_w^2\big),\phi_z\in L^2\big((0,T);H_w^2\big),\\ &\psi\in C\big([0,T];H^2\big),\psi_z\in L^2\big((0,T);H^2\big)\big\},\end{aligned}$$

where $w=1/U$. Notice that the traveling wave profile $U>0$ is bounded in $\mathbb{R}$. Then there is a constant $c_0>0$ such that $w\geq c_0$.

Define

$$N(t):=\sup_{\tau\in[0,t]}\big(\|\phi(\cdot,\tau)\|_{2,w}+\|\psi(\cdot,\tau)\|_2\big).$$

Using the Eq. (2.5a) and $V_z<0$, we obtain

$$\begin{aligned}&\frac{|U_z|}{U}=\frac{|V-s|}{d}\leq\frac{v_-}{d},\\ &\left|\left(\frac{\phi}{\sqrt{U}}\right)_z\right|=\left|\frac{\phi_z}{\sqrt{U}}-\frac{\phi U_z}{2U\sqrt{U}}\right|\leq\frac{|\phi_z|}{\sqrt{U}}+\frac{v_-}{2d}\cdot\frac{|\phi|}{\sqrt{U}}.\end{aligned} \tag{3.5}$$

Hence, by a Sobolev type inequality $\|f\|^2_{L^\infty} \le 2\|f\|_{L^2}\|f_x\|_{L^2}$ for all $f \in W^{1,2}(\mathbb{R})$, we have

$$\sup_{\tau\in[0,t]}\left\{\left\|\frac{\phi(\cdot,\tau)}{\sqrt{U}}\right\|_{L^\infty},\left\|\frac{\phi_z(\cdot,\tau)}{\sqrt{U}}\right\|_{L^\infty},\|\psi(\cdot,\tau)\|_{L^\infty},\|\psi_z(\cdot,\tau)\|_{L^\infty}\right\} \le C_0 N(t), \tag{3.6}$$

where $C_0 = C_0(v_-,d)$ is a positive constant. Owing to (3.2), Theorem 3.1 is a consequence of the following result for the reformulated system (3.4).

Proposition 3.1. *Let the assumptions in Theorem* 3.1 *hold. There exists a positive constant δ_0, such that if $N(0) \le \delta_0$, then the Cauchy problem* (3.4) *has a unique global solution $(\phi,\psi) \in X(0,+\infty)$ satisfying*

$$\begin{aligned}&\|\phi\|^2_{2,w} + \|\psi\|^2_2 + \int_0^t \left(\|\phi_z(\cdot,\tau)\|^2_{2,w} + \|\psi_z(\cdot,\tau)\|^2_2\right) d\tau \\ &\le C\left(\|\phi_0\|^2_{2,w} + \|\psi_0\|^2_2\right) \le CN^2(0)\end{aligned} \tag{3.7}$$

for all $t \in [0,+\infty)$. Moreover, it holds that

$$\sup_{z\in\mathbb{R}} |(\phi_z,\psi_z)(z,t)| \ \to\ 0 \quad as \quad t\ \to\ \infty. \tag{3.8}$$

To prove Proposition 3.1, we first present the local existence of a unique solution to system (3.4), which can be obtained by the standard iteration argument (cf. [4]) and details will be omitted for brevity.

Proposition 3.2 (Local Existence). *For any $\delta_1 > 0$, there exists a positive constant T_0 depending on δ_1 such that if $(\phi_0,\psi_0) \in H^2_w \times H^2$ and $N(0) \le \delta_1$, then* (3.4) *has a unique solution $(\phi,\psi) \in X(0,T_0)$ satisfying $N(t) \le 2N(0)$ for any $t \in [0,T_0]$.*

The global existence of solutions to system (3.4) follows from local existence of solutions stated in Proposition 3.2 alongside the following a priori estimates, based on the standard continuation argument.

Proposition 3.3 (A Priori Estimate). *Suppose that $(\phi,\psi) \in X(0,T)$ is a solution to* (3.4) *obtained in Proposition* 3.2 *for some $T > 0$. Then there exists a constant $\delta_2 > 0$ independent of T such that if $N(t) \le \delta_2$ for any $t \in [0,T]$, then the solution (ϕ,ψ) of* (3.4) *satisfies* (3.7) *for any $t \in [0,T]$.*

We prove Proposition 3.3 by a series of results shown below.

Lemma 3.1. *Under the assumptions of Proposition* 3.3*, there exists a constant $C > 0$ independent of t such that if $C_0 N(t) \le \min\{d,\varepsilon/3\}$, then the solution of* (3.4) *satisfies*

$$\int \frac{\phi^2}{U} + \int \psi^2 + d\int_0^t \int \frac{\phi_z^2}{U} + \varepsilon \int_0^t \int \psi_z^2 \le C \int \left(\frac{\phi_0^2}{U} + \psi_0^2\right). \tag{3.9}$$

Proof. Multiplying the Eq. (3.4a) by $2\phi/U$ and the Eq. (3.4b) by 2ψ, and adding them, we obtain

$$\begin{aligned}&\left(\frac{\phi^2}{U}+\psi^2\right)_t+\frac{2d\phi_z^2}{U}+2\varepsilon\psi_z^2+\phi^2\left[-\left(\frac{d}{U}\right)_{zz}+\left(\frac{s-V}{U}\right)_z\right]-2\sigma V_z\psi^2\\ =&\left[\frac{2d\phi\phi_z}{U}-\left(\frac{d}{U}\right)_z\phi^2+\frac{s-V}{U}\phi^2-2\phi\psi+2\varepsilon\psi_z\psi+(s-2\sigma V)\psi^2\right]_z\\ &-\frac{2\phi\phi_z\psi_z}{U}-2\sigma\psi\psi_z^2. \end{aligned}\tag{3.10}$$

By the Eq. (2.5a) , a direct calculation yields

$$-\left(\frac{d}{U}\right)_{zz}+\left(\frac{s-V}{U}\right)_z=0.\tag{3.11}$$

Then integrating (3.10) in z along with the fact $V_z<0$, we get

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{\phi^2}{U}+\psi^2\right)+2d\int\frac{\phi_z^2}{U}+2\varepsilon\int\psi_z^2+2\sigma\int|V_z|\psi^2\\ \le& 2\int\frac{|\phi\phi_z\psi_z|}{U}+2\sigma\int|\psi|\psi_z^2.\end{aligned}\tag{3.12}$$

By Young's inequality alongside (3.6) and the fact that $0<\sigma<1$, we can estimate the two terms on the right-hand side of (3.12) as follows:

$$\begin{aligned}&2\int\frac{|\phi\phi_z\psi_z|}{U}\le 2\left\|\frac{\phi}{\sqrt{U}}(\cdot,t)\right\|_{L^\infty}\int\frac{|\phi_z\psi_z|}{\sqrt{U}}\le C_0N(t)\int\frac{\phi_z^2}{U}+C_0N(t)\int\psi_z^2,\\ &2\sigma\int|\psi|\psi_z^2\le 2C_0N(t)\int\psi_z^2.\end{aligned}$$

Then substituting the above two inequalities into (3.12) yields that

$$\int\left(\frac{\phi^2}{U}+\psi^2\right)+\big(2d-C_0N(t)\big)\int_0^t\int\frac{\phi_z^2}{U}+\big(2\varepsilon-3C_0N(t)\big)\int_0^t\int\psi_z^2\le\int\left(\frac{\phi_0^2}{U}+\psi_0^2\right),$$

which gives (3.9) under our assumption $C_0N(t)\le\min\{d,\varepsilon/3\}$. □

We proceed to derive the estimate of the first order derivatives of (ϕ,ψ).

Lemma 3.2. *Let the assumptions of Proposition* 3.3 *hold. Then there exists a constant* $C>0$ *such that if* $C_0N(t)\le\min\{d,\varepsilon/3\}$, *then the solution of* (3.4) *satisfies*

$$\int\frac{\phi_z^2}{U}+\int\psi_z^2+d\int_0^t\int\frac{\phi_{zz}^2}{U}+\varepsilon\int_0^t\int\psi_{zz}^2\le C\int\left(\frac{\phi_{0z}^2}{U}+\frac{\phi_0^2}{U}+\psi_{0z}^2+\psi_0^2\right).\tag{3.13}$$

Proof. Differentiating (3.4) with respect to z yields

$$\begin{cases} \phi_{zt} = d\phi_{zzz} + (s-V)\phi_{zz} - V_z\phi_z - U\psi_{zz} - U_z\psi_z - (\phi_z\psi_z)_z, & (3.14a) \\ \psi_{zt} = \varepsilon\psi_{zzz} + (s-2\sigma V)\psi_{zz} - 2\sigma V_z\psi_z - \phi_{zz} - 2\sigma\psi_z\psi_{zz}. & (3.14b) \end{cases}$$

Multiplying the Eq. (3.14a) by $2\phi_z/U$ and the Eq. (3.14b) by $2\psi_z$, after some careful calculations, we end up with

$$\begin{aligned} &\left(\frac{\phi_z^2}{U} + \psi_z^2\right)_t + \frac{2d\phi_{zz}^2}{U} + 2\varepsilon\psi_{zz}^2 + \phi_z^2\left[-\left(\frac{d}{U}\right)_{zz} + \left(\frac{s-V}{U}\right)_z\right] \\ =&\left[\frac{2d\phi_{zz}\phi_z}{U} - \left(\frac{d}{U}\right)_z\phi_z^2 + \frac{(s-V)}{U}\phi_z^2 - 2\psi_z\phi_z + 2\varepsilon\psi_z\psi_{zz} + (s-2\sigma V)\psi_z^2\right]_z \\ &-\frac{2U_z\phi_z\psi_z}{U} - \frac{2V_z\phi_z^2}{U} - 2\sigma V_z\psi_z^2 - \frac{2\phi_z(\phi_z\psi_z)_z}{U} - 4\sigma\psi_z^2\psi_{zz}. \end{aligned} \tag{3.15}$$

Using (3.11) again, we integrate (3.15) over $\mathbb{R}\times(0,t)$ alongside the fact $\psi_z^2\psi_{zz} = (\psi_z^3/3)_z$ and arrive at

$$\begin{aligned} &\int\left(\frac{\phi_z^2}{U} + \psi_z^2\right) + 2d\int_0^t\int\frac{\phi_{zz}^2}{U} + 2\varepsilon\int_0^t\int\psi_{zz}^2 \\ =&\int\left(\frac{\phi_{0z}^2}{U} + \psi_{0z}^2\right) - 2\int\frac{U_z\phi_z\psi_z}{U} - 2\int\frac{V_z\phi_z^2}{U} - 2\int\sigma V_z\psi_z^2 - 2\int\frac{\phi_z(\phi_z\psi_z)_z}{U}. \end{aligned} \tag{3.16}$$

Recalling (3.5) and (2.5), we have

$$\frac{|U_z|}{U} = \frac{|V-s|}{d} \le \frac{v_-}{d}.$$

Thus, by Young's inequality, we get

$$\left|\frac{U_z\phi_z\psi_z}{U}\right| \le \frac{v_-|\phi_z\psi_z|}{d} \le \frac{d\phi_z^2}{2U} + \frac{v_-^2U\psi_z^2}{2d^3} \le \frac{d\phi_z^2}{2U} + \frac{\bar{u}v_-^2\psi_z^2}{2d^3},$$

where $\bar{u} = \sup_{z\in\mathbb{R}} U(z)$. Indeed, we have that $\bar{u} \le (1-\sigma)s^2$, where s varies in different wave profiles (see the statement of Theorem 2.1).

Using the Eq. (2.5b), one has

$$|V_z| \le \frac{\sigma v_-^2 + sv_- + (1-\sigma)s^2}{\varepsilon} =: c_0.$$

It then follows that

$$-\frac{2V_z\phi_z^2}{U} - 2\sigma V_z\psi_z^2 \le 2c_0\frac{\phi_z^2}{U} + 2\sigma c_0\psi_z^2.$$

Noticing that $\|\phi_z/\sqrt{U}\|_{L^\infty}\le C_0N(t)$ from (3.6), we use Cauchy-Schwarz inequality to get

$$\begin{aligned}\frac{\phi_z(\phi_z\psi_z)_z}{U}&=\frac{\phi_z}{\sqrt{U}}\cdot\frac{\phi_{zz}}{\sqrt{U}}\cdot\psi_z+\frac{\phi_z}{\sqrt{U}}\cdot\frac{\phi_z}{\sqrt{U}}\cdot\psi_{zz}\\&\le 2C_0N(t)\left(\frac{\phi_{zz}^2}{U}+\frac{\phi_z^2}{U}+\psi_z^2+\psi_{zz}^2\right).\end{aligned}$$

Then inserting the above estimates into (3.16) and using (3.9), we end up with

$$\begin{aligned}&\int\left(\frac{\phi_z^2}{U}+\psi_z^2\right)+2d\int_0^t\int\frac{\phi_{zz}^2}{U}+2\varepsilon\int_0^t\int\psi_{zz}^2\\\le&\int\left(\frac{\phi_{0z}^2}{U}+\psi_{0z}^2\right)+\left(C+C_0N(t)\right)\int_0^t\int\left(\frac{\phi_z^2}{U}+\psi_z^2\right)+C_0N(t)\int_0^t\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)\\\le&\, C\int\left(\frac{\phi_{0z}^2}{U}+\psi_{0z}^2+\frac{\phi_0^2}{U}+\psi_0^2\right)+C_0N(t)\int_0^t\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right),\end{aligned}\tag{3.17}$$

where we have used (3.6) in the second inequality. Therefore, if $C_0N(t)\le\min\{d,\varepsilon/2\}$, the inequality immediately (3.13) follows from (3.17). □

Next we give the estimates of the second order derivatives of (ϕ,ψ).

Lemma 3.3. *Under the assumptions of Proposition* 3.3*, there exists a constant* $C>0$ *such that if* $C_0N(t)\le\min\{d,\varepsilon/3\}$*, we have the following estimates:*

$$\begin{aligned}&\|\phi_{zz}\|_w^2+\|\psi_{zz}\|^2+d\int_0^t\|\phi_{zzz}(\cdot,\tau)\|_w^2d\tau+\varepsilon\int_0^t\|\psi_{zzz}(\cdot,\tau)\|^2d\tau\\\le&\, C\left(\|\phi_0\|_{2,w}^2+\|\psi_0\|_2^2\right),\end{aligned}\tag{3.18}$$

where $w=1/U$.

Proof. We differentiate (3.14) with respect to z to get

$$\begin{cases}\phi_{zzt}=d\phi_{zzzz}+(s-V)\phi_{zzz}-2V_z\phi_{zz}-V_{zz}\phi_z-U_{zz}\psi_z\\\qquad\quad-2U_z\psi_{zz}-U\psi_{zzz}-(\phi_z\psi_z)_{zz}, & \text{(3.19a)}\\\psi_{zzt}=\varepsilon\psi_{zzzz}+(s-2\sigma V)\psi_{zzz}-4\sigma V_z\psi_{zz}-2\sigma V_{zz}\psi_z-\phi_{zzz}-2\sigma(\psi_z\psi_{zz})_z. & \text{(3.19b)}\end{cases}$$

Multiplying the Eq. (3.19a) by $2\phi_{zz}/U$ and the Eq. (3.19b) by $2\psi_{zz}$, with some tedious computations, we have

$$\begin{aligned}&\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)_t+\frac{2d\phi_{zzz}^2}{U}+2\varepsilon\psi_{zzz}^2+\phi_{zz}^2\left[-\left(\frac{d}{U}\right)_{zz}+\left(\frac{s-V}{U}\right)_z\right]\\=&\left[\frac{2d\phi_{zzz}\phi_{zz}}{U}-\left(\frac{d}{U}\right)_z\phi_{zz}^2+\frac{(s-V)}{U}\phi_{zz}^2-2\psi_{zz}\phi_{zz}+2\varepsilon\psi_{zz}\psi_{zzz}+(s-2\sigma V)\psi_{zz}^2\right.\end{aligned}$$

$$
\left.-4\sigma\psi_z\psi_{zz}^2-\frac{2(\phi_z\psi_z)_z\phi_{zz}}{U}\right]_z-\frac{4V_z\phi_{zz}^2}{U}-\frac{2V_{zz}\phi_z\phi_{zz}}{U}-\frac{2U_{zz}\phi_{zz}\psi_z}{U}-\frac{4U_z\phi_{zz}\psi_{zz}}{U}
$$
$$
-6\sigma V_z\psi_{zz}^2-4\sigma V_{zz}\psi_z\psi_{zz}+2(\phi_z\psi_z)_z\left(\frac{\phi_{zz}}{U}\right)_z+4\sigma\psi_z\psi_{zz}\psi_{zzz}. \tag{3.20}
$$

Thanks to Eqs. (2.3) and (2.5) along with results in the existence Theorem 2.1, we can easily have

$$
|V_z|\le C,\quad |V_{zz}|\le C,\quad \frac{|U_{zz}|}{U}\le\frac{(V-s)^2}{d^2}+\frac{|V_z|}{d}\le C,\quad \frac{|U_z|}{U}\le\frac{v_-}{d} \tag{3.21}
$$

for some constant $C>0$. Then the following inequalities obviously hold:

$$
\left|\frac{4V_z\phi_{zz}^2}{U}\right|\le C\frac{\phi_{zz}^2}{U},\quad 6\sigma V_z\psi_{zz}^2\le C\psi_{zz}^2,\quad 4\sigma V_{zz}\psi_z\psi_{zz}\le C\left(\psi_z^2+\psi_{zz}^2\right).
$$

Using (3.6) and Cauchy-Schwarz inequality alongside the fact $0<\sigma<1$, we can get

$$
4\sigma\psi_z\psi_{zz}\psi_{zzz}\le 2C_0N(t)\left(\psi_{zz}^2+\psi_{zzz}^2\right)
$$

and

$$
\begin{aligned}
\left|(\phi_z\psi_z)_z\left(\frac{\phi_{zz}}{U}\right)_z\right| &\le\left|\psi_z\frac{\phi_{zz}}{\sqrt{U}}\frac{\phi_{zzz}}{\sqrt{U}}\right|+\left|\frac{v_-\psi_z}{d}\frac{\phi_{zz}^2}{U}\right|+\left|\psi_{zz}\frac{\phi_z}{\sqrt{U}}\frac{\phi_{zzz}}{\sqrt{U}}\right|+\left|\frac{v_-\psi_{zz}}{d}\frac{\phi_z}{\sqrt{U}}\frac{\phi_{zz}}{\sqrt{U}}\right|\\
&\le\frac{C\phi_z^2}{U}+\left(C+C_0N(t)\right)\frac{\phi_{zz}^2}{U}+2C_0N(t)\frac{\phi_{zzz}^2}{U}+C_0N(t)\psi_{zz}^2.
\end{aligned}
$$

Noticing that $U\ge 0$ is bounded, by Young's inequality and (3.21), one has

$$
\begin{aligned}
&\left|\frac{2V_{zz}\phi_z\phi_{zz}}{U}\right|\le C\left(\frac{\phi_z^2}{U}+\frac{\phi_{zz}^2}{U}\right),\\
&\frac{2|U_{zz}\phi_{zz}\psi_z|}{U}\le C|\phi_{zz}\psi_z|\le\frac{\phi_{zz}^2}{U}+C\psi_z^2,\\
&\frac{4|U_z\phi_{zz}\psi_{zz}|}{U}\le\frac{4v_-|\phi_{zz}\psi_{zz}|}{d}\le\frac{\phi_{zz}^2}{U}+C\psi_{zz}^2.
\end{aligned}
$$

Using the above inequalities alongside (3.11) and integrating (3.20) over $(0,t)\times(-\infty,\infty)$, we end up with

$$
\begin{aligned}
&\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)+2d\int_0^t\int\frac{\phi_{zzz}^2}{U}+2\varepsilon\int_0^t\int\psi_{zzz}^2\\
\le&\left(C+2C_0N(t)\right)\int_0^t\int\left(\frac{\phi_{zz}^2}{U}+\psi_{zz}^2\right)+C\int_0^t\int\left(\frac{\phi_z^2}{U}+\psi_z^2\right)\\
&+2C_0N(t)\int_0^t\int\left(\frac{\phi_{zzz}^2}{U}+\psi_{zzz}^2\right).
\end{aligned}
$$

Therefore, we get the desired inequality (3.18) by using (3.9) and (3.13) under the condition $C_0N(t)\le\min\{d,\varepsilon/3\}$. $\square$

Proof of Theorem 3.1. Owing to (3.2), Theorem 3.1 is a consequence of Proposition 3.1. We know the global estimate (3.7) has been given by Proposition 3.3 which is verified by the results in Lemmas 3.1-3.3. Therefore, to finish the proof of Theorem 3.1, it remains to prove (3.8) in view of (3.2). To show (3.8), we first recall a basic result (cf. [7]):

$$\text{if} \quad f\in W^{1,1}(0,\infty) \quad \text{and} \quad f\geq 0, \quad \text{then} \quad f \to 0 \quad \text{as} \quad t \to \infty. \tag{3.22}$$

Then from the global estimate (3.7), we claim that

$$\|\phi_z(\cdot,t)\|+\|\psi_z(\cdot,t)\| \to 0 \quad \text{as} \quad t \to +\infty. \tag{3.23}$$

We first show that $\|\phi_z(t)\|\to 0$ as $t\to+\infty$. Noticing that $\|\phi_z\|^2\in L^1(0,\infty)$ was given by (3.7) directly by the fact that $\|\phi_z\|^2\leq c\|\phi_z\|_w^2$ for some constant $c>0$, we just need to show $d(\|\phi_z\|^2)/dt\in L^1(0,\infty)$. To this end, we remark that U,V,U_z and V_z are bounded due to the results in Theorem 2.1 and equations in (2.5). Then from (3.14), (3.5) and (3.6), by the integration by parts and the Young inequality, we find

$$\begin{aligned}
\frac{d}{dt}\|\phi_z\|^2&=\frac{d}{dt}\int_{\mathbb{R}}\phi_z^2=2\int_{\mathbb{R}}\phi_z\phi_{zt}\\
&=2\int_{\mathbb{R}}\phi_z[d\phi_{zzz}+(s-V)\phi_{zz}-V_z\phi_z-U\psi_{zz}-U_z\psi_z-(\phi_z\psi_z)_z]\\
&\leq C\int_{\mathbb{R}}\left(\phi_z^2+\phi_{zz}^2+\psi_z^2+\psi_{zz}^2\right)\\
&\leq C\left(\|\phi_z(t)\|_{1,w}^2+\|\psi_z(t)\|_1^2\right),
\end{aligned}$$

where we have used the fact $\|\phi_z(t)\|_1\leq c_1\|\phi_z(t)\|_{1,w}$ for some constant $c_1>0$. Then it follows from the global estimate (3.7) that $d(\|\phi_z\|^2)/dt\in L^1(0,\infty)$, which further implies that $\|\phi_z(t)\|^2\to 0$ by (3.22) and hence $\|\phi_z(t)\|\to 0$ as $t\to+\infty$.

We next show that $\|\psi_z(t)\|\to 0$ as $t\to+\infty$. Since $\|\psi_z\|^2\in L^1(0,\infty)$ follows from (3.7) directly, it remains to show $d(\|\psi_z\|^2)/dt\in L^1(0,\infty)$. Similar to the argument above, we have

$$\begin{aligned}
\frac{d}{dt}\|\psi_z\|^2&=2\int_{\mathbb{R}}\psi_z\psi_{zt}\\
&=2\int_{\mathbb{R}}\psi_z[\varepsilon\psi_{zzz}+(s-2\sigma V)\psi_{zz}-2\sigma V_z\psi_z-\phi_{zz}-2\sigma\psi_z\psi_{zz}]\\
&\leq C\int_{\mathbb{R}}\left(\phi_{zz}^2+\psi_z^2+\psi_{zz}^2\right),
\end{aligned}$$

which alongside (3.7) implies $d(\|\psi_z\|^2)/dt\in L^1(0,\infty)$. This gives that $\|\psi_z(t)\|^2\to 0$ due to (3.22) and hence $\|\psi_z(t)\|\to 0$ as $t\to\infty$.

Now with (3.23) in hand, for all $z\in\mathbb{R}$, it follows from (3.7) that

$$\begin{aligned}
\phi_z^2(z,t)&=2\int_{-\infty}^{z}\phi_z\phi_{zz}(y,t)dy\\
&\leq 2\|\phi_z(t)\|\|\phi_{zz}(t)\|\leq 2\|\phi_z(t)\|\|\phi_{zz}(t)\|_w \to 0 \quad \text{as} \quad t \to \infty.
\end{aligned}$$

This implies that $\sup_{z\in\mathbb{R}}|\phi_z(z,t)|\to 0$ as $t\to\infty$. The same argument applied to ψ_z gives

$$\sup_{z\in\mathbb{R}}|\psi_z(z,t)| \;\to\; 0 \quad \text{as} \quad t \;\to\; \infty.$$

Hence, (3.8) is proved and the proof of Proposition 3.1 is finished. Consequently, we obtain Theorem 3.1. $\square$

4 Summary and discussion

We are concerned with the existence and nonlinear stability of a conservational system (1.3) with a parameter $\sigma=-\varepsilon/\chi$, which is transformed from a singular Keller-Segel type chemotaxis system (1.1) via the Cole-Hopf transformation (1.2). When $\sigma<0$ (i.e. $\chi>0$ and chemotaxis is attractive), it has been shown in the literature [19,23] that the system (1.3) admits a unique type of traveling wave solution $(U,V)(z)$ satisfying $U_z<0$ and $V_z>0$, which is nonlinearly stable against a small perturbation. In this paper, we consider the traveling wave solutions of (1.3) with $\sigma>0$ (i.e. $\chi<0$ and chemotaxis is repulsive) and surprisingly find that there are three different types of traveling wave profiles (U,V), in contrast to the case $\sigma<0$. The common feature of these three different traveling wave profiles is that $V_z<0$ (namely V is always monotonically decreasing), see Theorem 2.1 and numerical simulations showing different traveling wave profiles in Figs. 3-5. Our results indicate the repulsive chemotaxis may have very different dynamics from attractive chemotaxis. Moreover, we develop a unified approach based on the weighted energy estimates and the technique of taking anti-derivative to show that all traveling wave solutions are nonlinearly stable if the initial data are small perturbations with zero integral from the spatially shifted traveling wave profiles (see Theorem 3.1).

In the rest of this section, we shall discuss the existence and/or stability of traveling wave solutions to the original Keller-Segel system (1.1). Due to the possible logarithmic singularity, the Keller-Segel system (1.1) was rarely directly studied. Instead it was often studied via the Cole-Hopf transformation (1.2) which transforms (1.1) into a nonsingular system (1.3). Now a natural question is whether the results of the transformed system (1.3) can be transferred to the original Keller-Segel system (1.1). The answer seems to be elusive since the Cole-Hopf transformation (1.2) is not necessarily solvable for w given v solving (1.3). Below we shall briefly discuss whether a traveling wave solution of (1.1) can be obtained from (1.3) via (1.2).

In the attractive case $\chi>0,\mu>0$, where $\sigma=-\varepsilon/\chi<0$, it was shown (cf. [32]) that (1.3) admits a unique traveling wave solution (U,V) such that $V<0,U_z<0,V_z>0$. Then solving the Cole-Hopf transformation $V=\sqrt{\chi\mu}W_z/(\mu W)$ for W, one can get a traveling wave profile $W(z)$ satisfying $W_z>0$, which along with U gives a traveling wave solution of (1.1), see details in [24]. However, in the repulsive case $\chi<0,\mu<0$ considered in this paper, where $\sigma>0$, we have shown that (1.3) admits three different traveling wave profiles (U,V) as recorded in Theorem 2.1. Now assuming $(U,W)(z)$ is a traveling wave solution of (1.1) satisfying $(U,W)(\pm\infty)=(u_\pm,w_\pm)$, where W is solved from (1.2)

$$\lambda V(z)=-\frac{W_z}{W} \tag{4.1}$$

with $\lambda=-\mu/\sqrt{\chi\mu}>0$. Then there must exist a point $z_*\in\mathbb{R}$ such that $W(z_*)\neq 0$ (since otherwise $W\equiv 0$ is not a traveling wave profile). Solving (4.1) directly, we get

$$W(z)=W(z_*)e^{-\lambda\int_{z_*}^{z}V(\xi)d\xi}. \tag{4.2}$$

Note that the traveling wave profile (U,V) of (1.3) satisfies $V>0$ and $V_z<0$ (see the statement of Theorem 2.1), namely V is a monotonically decreasing traveling wavefront with $V(-\infty)=v_->0$. Then it follows that $\int_{z_*}^{z}V(\xi)d\xi\to-\infty$ as $z\to-\infty$. This immediately implies by sending $z\to-\infty$ in (4.2)

$$W(-\infty)=\infty,$$

which contradicts the fact $W(-\infty)=w_-$. That is the function $W(z)$ in (4.2) solved from (4.1) does not give a traveling wave profile for (1.1) although U is a traveling wave profile of (1.1). This exhibits an interesting phenomenon: in a two-component repulsive chemotaxis model (1.1), one solution component is a traveling wave profile but the other is not. This differs from the attractive chemotaxis model (1.1), which has been shown (cf. [24]) to admit both traveling wave profiles U and W via the Cole-Hopf transformation (1.2).

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