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# A Parallel, High-Order Direct Discontinuous Galerkin Method for the Navier-Stokes Equations on 3D Hybrid Grids

Jian Cheng[1,*], Xiaodong Liu[2], Tiegang Liu[1] and Hong Luo[2]

[1] *School of Mathematics and Systems Science, Beihang University, Beijing 100191, P.R. China.*
[2] *Department of Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, NC, 27695, USA.*

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**Abstract.** A parallel, high-order direct Discontinuous Galerkin (DDG) method has been developed for solving the three dimensional compressible Navier-Stokes equations on 3D hybrid grids. The most distinguishing and attractive feature of DDG method lies in its simplicity in formulation and efficiency in computational cost. The formulation of the DDG discretization for 3D Navier-Stokes equations is detailed studied and the definition of characteristic length is also carefully examined and evaluated based on 3D hybrid grids. Accuracy studies are performed to numerically verify the order of accuracy using flow problems with analytical solutions. The capability in handling curved boundary geometry is also demonstrated. Furthermore, an SPMD (single program, multiple data) programming paradigm based on MPI is proposed to achieve parallelism. The numerical results obtained indicate that the DDG method can achieve the designed order of accuracy and is able to deliver comparable results as the widely used BR2 scheme, clearly demonstrating that the DDG method provides an attractive alternative for solving the 3D compressible Navier-Stokes equations.

**AMS subject classifications**: 65M60, 65M99, 35L65

**Key words**: Direct discontinuous Galerkin method, compressible Navier-Stokes equations, hybrid grids.

## 1 Introduction

Discontinuous Galerkin (DG) methods [1–4], as a typical representative in the community of high order methods, have been widely used in computational fluid dynamics,

*Corresponding author. *Email addresses:* chengjian@buaa.edu.cn (J. Cheng), xliu29@ncsu.edu (X. Liu), liutg@buaa.edu.cn (T. Liu), hong_luo@ncsu.edu (H. Luo)

 

computational acoustics, and computational magneto-hydrodynamics. The reason why DG methods have been intensively studied and widely applied is because the various attractive features they possess: (1) They have several useful mathematical properties with respect to conservation, stability, and convergence; (2) The method can be easily extended to higher-order ($> 2^{nd}$) approximation; (3) The methods are well suited for complex geometries since they can be applied on unstructured grids. In addition, the methods can also handle non-conforming elements, where the grids are allowed to have hanging nodes; (4) The methods are highly parallelizable, as they are compact, thus, the inter-element communications are minimal, domain decomposition can be efficiently employed. The compactness also allows for structured and simplified coding for the methods; (5) They can easily handle adaptive strategies, since refining or coarsening a grid can be achieved without considering the continuity restriction commonly associated with the conforming elements. The methods allow easy implementation of hp-refinement, for example, the order of accuracy, or shape, can vary from element to element; (6) They have the ability to compute low Mach number flow problems without recourse to the time-preconditioning techniques normally required by the finite volume schemes.

DG methods are indeed a natural choice for solving hyperbolic equations, such as the compressible Euler equations. However, the DG formulation is far less certain and advantageous for diffusion problems, such as the compressible Navier-Stokes equations, where viscous and heat fluxes exist and require the evaluation of the solution derivatives at the cell interface. Taking a simple arithmetic mean of the solution derivatives from the left and right is inconsistent, because it does not take into account the underlying jumps of DG solutions at the cell interface. A number of methods have been proposed in the literature to address this issue. Several of them are listed in the following table.

Table 1: Different discontinuous Galerkin methods for diffusion problems.

| Method | Developed by |
|---|---|
| interior penalty(IP) | Arnold et al. |
| symmetric IP method(SIP) | Hartmann et al. |
| local DG method(LDG) | Cockburn and Shu |
| compact DG method(CDG) | Peraire and Persson |
| hybridizable DG method(HDG) | Cockburn et al. |
| Bassi-Rebay method(BR1/BR2) | Bassi and Rebay |
| reconstructed DG(rDG) | Luo et al. |
| recovery DG(RDG) | van Leer et al. |
| direct DG method(DDG) | Liu and Yan |

The first attempt at using DG methods to solve elliptic and parabolic problems can be tracked back to the late 1970s and early 1980s when an interior penalty (IP) method was independently proposed and studied in [18–20]. In the IP method, a viscous flux is obtained through the average of the left and right states and then penalizing with