A Numerical Analysis of the Weak Galerkin Method for the Helmholtz Equation with High Wave Number

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Abstract. We study the error analysis of the weak Galerkin finite element method in [24, 38] (WG-FEM) for the Helmholtz problem with large wave number in two and three dimensions. Using a modified duality argument proposed by Zhu and Wu, we obtain the pre-asymptotic error estimates of the WG-FEM. In particular, the error estimates with explicit dependence on the wave number k are derived. This shows that the pollution error in the broken H^1-norm is bounded by $\mathcal{O}(k(kh)^{2p})$ under mesh condition $k^{7/2}h^2 \leq C_0$ or $(kh)^2 + k(kh)^{p+1} \leq C_0$, which coincides with the phase error of the finite element method obtained by existent dispersion analyses. Here h is the mesh size, p is the order of the approximation space and C_0 is a constant independent of k and h. Furthermore, numerical tests are provided to verify the theoretical findings and to illustrate the great capability of the WG-FEM in reducing the pollution effect.

1 Introduction

Let $\Omega \in \mathbb{R}^d, d = 2,3$, be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. We consider the following Helmholtz problem with the Robin boundary condition:

$$-\Delta u - k^2 u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} + \mathbf{i}ku = g \qquad \text{on } \Gamma, \tag{1.2}$$

where $\mathbf{i} = \sqrt{-1}$ denotes the imaginary unit and $\mathbf{n}$ denotes the unit outward normal to Γ.

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The above Helmholtz problem is an approximation of the acoustic scattering problem. The Robin boundary condition (1.2) is known as the first order approximation of the radiation condition [13]. We remark that the Helmholtz problem (1.1)-(1.2) also arises in applications as a consequence of frequency domain treatment of attenuated scalar waves [10].

It is well-known that the finite element method of fixed order for the Helmholtz problem (1.1)-(1.2) at high frequencies ($k \gg 1$) is subject to the effect of pollution: the ratio of the error of the finite element solution to the error of the best approximation from the finite element space cannot be uniformly bounded with respect to k [1,3,4,9,16,18,19]. In other words, the error bound of the finite element solution to the Helmholtz problem (1.1)-(1.2) usually consists of two parts: one is the same order as the error of the best approximation of u from the finite element space, the other dominates the error bound of the finite element solution for large wave number k. The second part is the so-called pollution error (cf. [8,17]). We recall that, the term "asymptotic error estimate" refers to the error estimate without pollution error and the term "preasymptotic error estimate" refers to the estimate with non-negligible pollution effect.

However, the highly indefinite nature of the Helmholtz problem with high wave number makes the error analysis of the FEM (including discontinuous Galerkin methods) very difficult. The reader is referred to [21,22] for the pollution free error estimates of the FEM for the one and higher dimensional Helmholtz problems, and to [11,12,33,34] for the estimates with pollution error of the FEM and CIP-FEM for two and three dimensional Helmholtz problem.

Weak Galerkin finite element methods were first introduced as nonconforming methods in [31] by Wang and Ye in 2013 for second order elliptic equations, which has been used to solve various problems [36–38]. The WG-FEMs admit various finite element meshes, such as a mix of arbitrary shape of polygons and polyhedrons and less number of the degree of freedoms in algebraic system than the general DG methods after parallel computation. The biggest feature of WG-FEMs is their ability to replace the classic derivatives in various variational formulations by the weak derivatives defined in [31]. Wang and Ye have applied the WG formulation for solving the Helmholtz problem in [38] and have shown the error estimates under the mesh condition $k^2 h \le C_0$ by using the Schatz argument, where C_0 is a constant independent of the wave number k and the mesh size h of a uniform partition. Since $k^2 h \le C_0$ is too strict for large k, later they improved the mesh condition and derived the stability and well-posedness without a mesh size constraint using arguments similar to those provided in [14,15] by Feng and Wu. However, the convergence rate in their estimates lost one order under the general assumption that $kh \le 1$ for the large wave number problem. The goal of this work is to obtain the optimal pre-asymptotic error estimates under the mesh condition $k^{7/2} h^2 \le C_0$ or $(kh)^2 + k(kh)^{p+1} \le C_0$ by using a modified dual argument like the one recently proposed in [33,34].

Other than the WG-FEMs, various discontinuous Galerkin methods are also nonconforming methods. We refer the reader to [20] by Melenk et al. for the latest asymptotic