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Characteristic Local Discontinuous Galerkin Methods for Incompressible Navier-Stokes Equations

Shuqin Wang[1,2], Weihua Deng[1,*], Jinyun Yuan[2] and Yujiang Wu[1]

[1] *School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China.*
[2] *Department of Mathematics, Federal University of Paraná, Centro Politécnico, CP: 19.081, Curitiba, CEP: 81531-990, PR, Brazil.*

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Abstract. By combining the characteristic method and the local discontinuous Galerkin method with carefully constructing numerical fluxes, variational formulations are established for time-dependent incompressible Navier-Stokes equations in $\mathbb{R}^2$. The nonlinear stability is proved for the proposed symmetric variational formulation. Moreover, for general triangulations the priori estimates for the L^2-norm of the errors in both velocity and pressure are derived. Some numerical experiments are performed to verify theoretical results.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Navier-Stokes equations, local discontinuous Galerkin method, symmetric variational formulation.

1 Introduction

Based on the assumption that the fluid, at the scale of interest, is a continuum, and the conservation of momentum (often alongside mass and energy conservation), the equation to describe the motion of fluid substances can be derived, which is named after the French engineer and physicist Claude-Louis Navier and the Ireland mathematician and physicist George Gabriel Stokes to recognize their fundamental contributions. Nowadays, it is still the central equation to fluid mechanics. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary $\partial\Omega$ and $T>0$ is a finite quantity. The

*Corresponding author. *Email addresses:* `wsqlzu@gmail.com` (S. Wang), `dengwh@lzu.edu.cn` (W. Deng), `yuanjy@gmail.com` (J. Yuan), `myjaw@lzu.edu.cn` (Y. Wu)

time-dependent Navier-Stokes equation for an incompressible viscous fluid confined in Ω is [27]:

$$\begin{cases} \partial_t \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f}, & (\boldsymbol{x},t)\in\Omega\times[0,T], \\ \nabla\cdot\boldsymbol{u}=0, & (\boldsymbol{x},t)\in\Omega\times[0,T], \\ \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \\ \boldsymbol{u}(\boldsymbol{x},t)=0, & (\boldsymbol{x},t)\in\partial\Omega\times[0,T]. \end{cases} \tag{1.1}$$

It is well-known that the problem has a unique solution and $\boldsymbol{u}\in L^2(0,T;H_0^1(\Omega)^2)\cap L^\infty(0,T;L^2(\Omega)^2)$, $p\in W^{-1,\infty}(0,T;L_0^2(\Omega))$ for $\boldsymbol{u}_t\in L^2(0,T;\boldsymbol{X}')$, the body force function $\boldsymbol{f}\in L^2(0,T;H^{-1}(\Omega)^2)$ and $\boldsymbol{u}_0\in H(\mathrm{div},\Omega)$ [27]. The constant ν is the fluid viscosity coefficient. Since p is uniquely defined up to an additive constant, we also assume that $\int_\Omega p\,d\boldsymbol{x}=0$. The $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is a nonlinear convective term and

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=u_1\frac{\partial\boldsymbol{u}}{\partial x}+u_2\frac{\partial\boldsymbol{u}}{\partial y}.$$

The idea of the characteristic methods dates back to the works of Douglas and Russell in 1982 [15]. Later on Süli [26] and Boukir et al. [4] extended the idea to two and three dimensional nonlinear coupled system, and performed the detailed numerical analysis for the incompressible Navier-Stokes equation. In the context of linear advection-diffusion equations, Eulerian-Lagrangian characteristic methods were proved to converge independent of the vanishing viscosity parameter [28, 29] or even in the case of degenerate diffusion coefficient [32]. The Eulerian-Lagrangian characteristic method was also used to solve the equation modelling the subsurface porous medium flow with error estimate [30]. Being different from the above ideas, here we use the characteristic method to tackle the time derivative term and the nonlinear convective term together and solve the considered equation with first order accuracy in time. It seems that the characteristic methods have many advantages compared to a high-order Runge-Kutta scheme or a high-order finite difference scheme [14], such as 1) efficient in solving the advection-dominated diffusion problems; 2) easily obtaining the existence and uniqueness of the solutions of the discretized system; 3) making the nonlinear equations linear and conveniently tackling the nonlinear obstacles; 4) easily performing the numerical stability analysis; 5) physically discretizing the material derivative [8].

Because of the inherent performances of the Navier-Stokes or Stokes equations in characterizing the turbulence (most flows occurring in nature are turbulent) in fluids or gases, from the finite element methods to discontinuous Galerkin methods a lot of research works on these topics have been done [3, 9–12, 17–19, 21, 24]. To our knowledge, there are less works on the discontinuous Galerkin method to solve the time-dependent incompressible Navier-Stokes equation, and much less on the local discontinuous Galerkin method (LDG). Recently splitting the nonlinearity and incompressibility, and using discontinuous or continuous finite element methods in space, Girault et al. solved the time-dependent incompressible Navier-Stokes equation [17] with discontinuous Galerkin methods.