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Robust Globally Divergence-Free Weak Galerkin Finite Element Methods for Natural Convection Problems

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Abstract. This paper proposes and analyzes a class of weak Galerkin (WG) finite element methods for stationary natural convection problems in two and three dimensions. We use piecewise polynomials of degrees $k,k-1$, and k $(k\geq 1)$ for the velocity, pressure, and temperature approximations in the interior of elements, respectively, and piecewise polynomials of degrees l,k,l $(l=k-1,k)$ for the numerical traces of velocity, pressure and temperature on the interfaces of elements. The methods yield globally divergence-free velocity solutions. Well-posedness of the discrete scheme is established, optimal a priori error estimates are derived, and an unconditionally convergent iteration algorithm is presented. Numerical experiments confirm the theoretical results and show the robustness of the methods with respect to Rayleigh number.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Natural convection, Weak Galerkin method, Globally divergence-free, error estimate, Rayleigh number.

1 Introduction

Let $\mathbb{R}^d (d=2,3)$ be a polygonal or polyhedral domain with a polygonal or polyhedral subdomain $\Omega_f\subset\Omega$ and $\Omega_s:=\Omega\backslash\Omega_f$, we consider the following stationary natural convection (or conduction-convection) problem: seek the velocity $\boldsymbol{u}=(u_1,u_2,\cdots,u_d)^T$, the pressure p, and the temperature T such that

$$\begin{cases} -\mathrm{Pr}\Delta\boldsymbol{u}+\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p-\mathrm{Pr}Ra\boldsymbol{j}T=\boldsymbol{f} & \text{in } \Omega_f, \\ \nabla\cdot\boldsymbol{u}=0 & \text{in } \Omega_f, \\ -\kappa\Delta T+\nabla\cdot(\boldsymbol{u}T)=g & \text{in } \Omega, \\ \boldsymbol{u}\equiv\boldsymbol{0} & \text{in } \Omega_s\bigcup\partial\Omega_f, \\ T=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\otimes$ is defined by $\boldsymbol{u}\otimes\boldsymbol{v}=(u_iv_j)_{d\times d}$ for $\boldsymbol{v}=(v_1,v_2,\cdots,v_d)^T$, $\boldsymbol{j}$ is the vector of gravitational acceleration with $\boldsymbol{j}=(0,1)^T$ when $d=2$ and $\boldsymbol{j}=(0,0,1)^T$ when $d=3$, $\boldsymbol{f}\in[L^2(\Omega_f)]^d$, $g\in L^2(\Omega)$ are the forcing functions, and Pr, Ra denote the Prandtl and Rayleigh numbers, respectively,.

The model problem (1.1), arising both in nature and in engineering applications, is a coupled system of fluid flow, governed by the incompressible Navier-Stokes equations, and heat transfer, governed by the energy equation. Due to its practical significance, the development of efficient numerical methods for natural convection has attracted a great many of research efforts; see, e.g. [1–3,12,15,18–23,25,26,28,29,38,39]. In [2,3], error estimates for some finite element methods were derived in approximating stationary and non-stationary natural convection problems. [19, 20] applied Petrov-Galerkin least squares mixed finite element methods to discretize the problems. [25, 26] developed a nonconforming mixed element method and a Petrov-Galerkin least squares nonconforming mixed element method for the stationary problems. In [37], three kinds of decoupled two level finite element methods were presented. [38, 39] applied the variational multiscale method to solve the stationary and non-stationary problems.

In this paper, we consider a weak Galerkin (WG) finite element discretization of the model problem (1.1). The WG method was first proposed and analyzed to solve second-order elliptic problems [30, 31]. It is designed by using a weakly defined gradient operator over functions with discontinuity, and then allows the use of totally discontinuous functions in the finite element procedure. Similar to the hybridized discontinuous Galerkin (HDG) method [11], the WG method is of the property of local elimination of unknowns defined in the interior of elements. We note that in some special cases the WG method and the HDG method are equivalent (cf. [6–8]). In [6], a class of robust globally divergence-free weak Galerkin methods for Stokes equations were developed, and then were extended in [40] to solve incompressible quasi-Newtonian Stokes equations.We also refer to [9,10,13,16,17,24,32–36,41] for some other developments and applications of the WG method.

This paper aims to propose a class of WG methods for the natural convection problems. The methods include as unknowns the velocity, pressure, and temperature variables both in the interior of elements and on the interfaces of elements. In the interior of elements, we use piecewise polynomials of degrees $k,k-1$, and k $(k\geq 1)$ for the velocity, pressure, and temperature approximations, respectively. On the interfaces of elements, we use piecewise polynomials of degrees l,k,l $(l=k-1,k)$ for the numerical traces of velocity, pressure and temperature. The methods are shown to yield globally divergence-free velocity approximations.

The rest of the paper is organized as follows. Section 2 introduces the WG finite element scheme. Section 3 shows the existence and uniqueness of the discrete solution. Section 4 derives a priori error estimates. Section 5 discusses the local elimination property and the convergence of an iteration method for the WG scheme. Finally, Section 6 provides numerical examples to verify the theoretical results.