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# Efficient, Accurate and Energy Stable Discontinuous Galerkin Methods for Phase Field Models of Two-Phase Incompressible Flows

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**Abstract.** The goal of this paper is to propose fully discrete local discontinuous Galerkin (LDG) finite element methods for the Cahn-Hilliard-Navier-Stokes (CHNS) equation, which are shown to be unconditionally energy stable. In details, using the convex splitting principle, we first construct a first order scheme and a second order Crank-Nicolson scheme for time discretizations. The proposed schemes are shown to be unconditionally energy stable. Then, using the invariant energy quadratization (IEQ) approach, we develop a novel linear and decoupled first order scheme, which is easy to implement and energy stable. In addition, a semi-implicit spectral deferred correction (SDC) method combining with the first order convex splitting scheme is employed to improve the temporal accuracy. Due to the local properties of the LDG methods, the resulting algebraic equations at the implicit level is easy to implement and can be solved in an explicit way when it is coupled with iterative methods. In particular, we present efficient and practical multigrid solvers to solve the resulting algebraic equations, which have nearly optimal complexity. Numerical experiments of the accuracy and long time simulations are presented to illustrate the high order accuracy in both time and space, the capability and efficiency of the proposed methods.

**AMS subject classifications**: 65M60, 35Q35, 35Q30, 35G25

**Key words**: Cahn-Hilliard-Navier-Stokes equation, local discontinuous Galerkin method, convex splitting, invariant energy quadratization, unconditional energy stability, spectral deferred correction method.

## 1 Introduction

The Cahn-Hilliard-Navier-Stokes (CHNS) phase field model arises as a diffuse interface model for the flow of two immiscible and incompressible fluids [1]. Let $\Omega$ be a bounded

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convex polygonal domain in $\mathbb{R}^d (d \le 3)$. The non-dimensional CHNS equation takes the following form

$$\begin{cases} \phi_t = \nabla\cdot(M(\phi)\nabla\mu) - \nabla\cdot(\phi\boldsymbol{u}), & \text{in } \Omega_T, \\ \mu = F'(\phi) - \varepsilon^2\Delta\phi, & \text{in } \Omega_T, \\ \boldsymbol{u}_t - \dfrac{1}{Re}\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = -\dfrac{\varepsilon^{-1}}{We^*}\phi\nabla\mu, & \text{in } \Omega_T, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega_T, \end{cases} \tag{1.1}$$

where $\Omega_T = \Omega\times(0,T)$, $Re$ is the Reynolds number, $We^*$ is the modified Weber number that measures the relative rates of diffusion and the interface parameter $\varepsilon$ determines the width of the interface separating the two fluids (the interface is of width $\mathcal{O}(\varepsilon)$). $M(\phi)\ge 0$ is a mobility that incorporates the Peclet number, $\boldsymbol{u}$ is the advective velocity, $p$ is the modified pressure, $\mu$ is the chemical potential, $\phi$ is the phase field variable and $F(\phi) = \frac{1}{4}(1-\phi^2)^2$ is the double well potential.

Throughout the paper, we assume that Eq. (1.1) is supplemented with the following Dirichlet boundary conditions for the velocity $\boldsymbol{u}$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \partial\Omega\times(0,T), \tag{1.2}$$

and Neumann boundary conditions for $\phi$ and $\mu$

$$\nabla\phi\cdot\boldsymbol{n} = \nabla\mu\cdot\boldsymbol{n} = 0, \quad \text{on } \partial\Omega\times(0,T), \tag{1.3}$$

where $\boldsymbol{n}$ is the unit normal vector on $\partial\Omega$ pointing exterior to $\Omega$, or the periodic boundary conditions for all variables. Under the above boundary conditions, the CHNS equation (1.1) is mass conservative and energy dissipative, namely

$$\frac{d}{dt}E = -\frac{1}{Re}\int_\Omega |\nabla\boldsymbol{u}|^2 d\boldsymbol{x} - \frac{\varepsilon^{-1}}{We^*}\int_\Omega M(\phi)|\nabla\mu|^2 d\boldsymbol{x} \le 0, \tag{1.4}$$

where the energy functional $E$ is defined as

$$E = \int_\Omega \frac{1}{2}|\boldsymbol{u}|^2 d\boldsymbol{x} + \frac{1}{We^*}\int_\Omega \left(\frac{1}{\varepsilon}F(\phi) + \frac{\varepsilon}{2}|\nabla\phi|^2\right) d\boldsymbol{x}. \tag{1.5}$$

The energy dissipation law (1.4) is crucial and serves as a guide for the design of numerical schemes. From the numerical point of view, people are particularly interested in designing numerical schemes that preserve the corresponding energy stability result.

There have been many algorithms developed and simulations performed for the Cahn-Hilliard equation [4, 12, 23, 30], also for the CHNS equation, for example, the first order [2, 11, 21, 22, 24, 26] and second order [18–20, 26] temporal discretization schemes, in the framework of finite element methods [28]. Specially, Shen and Yang [27] constructed two classes of decoupled and unconditionally energy stable schemes for Cahn-Hilliard phase-field models of two-phase incompressible flows, based on stabilization and convex