Commun. Comput. Phys.
doi: 10.4208/cicp.2019.js60.02

Vol. **26**, No. 5, pp. 1397-1414
November 2019

Numerical Analysis of the Midpoint Scheme for the Generalized Benjamin-Bona-Mahony Equation with White Noise Dispersion

Guillaume Fenger, Olivier Goubet* and Youcef Mammeri

LAMFA CNRS UMR 7352, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens, France.

Received 5 February 2019; Accepted (in revised version) 6 May 2019

Abstract. We consider a midpoint scheme to approximate analytical solutions to a white noise driven BBM equation that reads

$$du-du_{xx}+u_x\circ dW+u^p u_x dt=0.$$

We prove the well-posedness of the time-discrete approximation scheme and we provide the strong error order, which is 1.

AMS subject classifications: 35Q53, 65C30, 65N12, 65N15

Key words: Stochastic long wave equations, midpoint scheme, strong order of convergence.

1 Introduction

1.1 Dispersive equations with white noise modulation

The regularized long wave equation, also known as Benjamin-Bona-Mahony (BBM) equation has been introduced as an alternate model to Korteweg-de Vries equation for the propagation of one-way long wave in shallow water. For general nonlinearities, the equation (gBBM) reads

$$u_t-u_{txx}+u_x+u^p u_x=0. \tag{1.1}$$

These deterministic equations have been widely studied theoretically and numerically in the mathematical literature; see for instance the initial value problem in [6,7,22], the decay rate of solutions for small initial data in [1,2,28] or numerical computations in [3,21].

*Corresponding author. *Email addresses:* `guillaume.fenger@u-picardie.fr` (G. Fenger), `olivier.goubet@u-picardie.fr` (O. Goubet), `youcef.mammeri@u-picardie.fr` (Y. Mammeri)

Recent years have seen an ever increasing study about stochastic dispersive equations, whether for theoretical [8–10, 13–17, 20] or numerical aspects [5, 11, 18, 19, 26]. Here we are interested in gBBM equations driven by white noise modulation. Quantities such as the bathymetry, the air pressure on the surface, or the wind effect, are stochastic in nature, it is therefore desirable to study stochastic shallow water wave equations [12]. We are dealing with the stochastic model studied in [8]. Here $x \in \mathbb{T}$ the one-dimension torus.

The study of dispersive equations with white noise modulation has latterly started with nonlinear Schrödinger equations (NLS), see [13, 20]; NLS equations model also the propagation of water waves but in deep water. Such investigation furthers the study where the dispersion is driven by a deterministic varying function [4]. Since the dispersive solutions are oscillating, choosing the appropriate method to solve this type of stochastic PDE is of great interest, the approximation being generally of a lower order than in the deterministic case [24, 25]. Typically, time derivatives of dispersive PDEs are discretized by semi-implicit schemes [5], splitting methods [26], or exponential integrators [11]. These schemes have the advantage of solving the linear part unconditionally stable. Nonetheless, Euler-Maruyama scheme has been applied for the NLS equation with multiplicative noise in [14]. Authors showed that the strong error order, i.e. the mean-square order [24, 25], is $1/2$, while the weak order, i.e. in the distribution sense [27, 30], is 1.

In [5], the authors provide the strong error order for the Crank-Nicolson scheme approximating the NLS with white noise modulation. They proved that the strong order is 1 instead of 2 in the deterministic case. In this work we prove that for modulated stochastic BBM equation we also have a strong order 1 for a midpoint scheme; this is expected since, like the Crank-Nicolson scheme, the midpoint scheme has order 2 for deterministic ODE. It is worth to point out that the arguments used for NLS and for BBM are different.

The article is organized as follows. We complete the introduction reminding some important results about the generalized BBM equation with white noise modulation and we set the mathematical framework. In Section 2 the numerical scheme is presented and the main results are stated. In Sections 3 and 4, we provide the proofs of the main theorems. We discuss some numerical computations in Section 5.

1.2 A BBM equation with white noise modulation

We address the generalized BBM equation with white noise dispersion introduced in [8]. This equation reads, for p integer ≥ 1, $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the one-dimensional torus.

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0 \tag{1.2}$$

in Stratonovich's formulation. This stands for the stochastic differential equation in $H^1(\mathbb{T})$

$$du + Au \circ dW + AF(u)dt = 0, \tag{1.3}$$

where A is the bounded skew symmetric operator $\partial_x(1-\partial_x^2)^{-1}$, the nonlinear term reads $F(u) = \frac{u^{p+1}}{p+1}$, and $W(t)$ is a standard real valued Brownian motion. Eq. (1.3) is a short-hand