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Müntz Spectral Method for Two-Dimensional Space-Fractional Convection-Diffusion Equation

Dianming Hou[1,2], Mejdi Azaiez[3] and Chuanju Xu[2,*]

[1] *School of Mathematics and Statistics, Jiangsu Normal University, 221116 Xuzhou, China.*
[2] *School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High Performance Scientific Computing, Xiamen University, 361005 Xiamen, China.*
[3] *Bordeaux INP, Laboratoire I2M UMR 5295, 33607 Pessac, France.*

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Abstract. In this paper, we propose and analyze a Müntz spectral method for a class of two-dimensional space-fractional convection-diffusion equations. The proposed methods make new use of the fractional polynomials, also known as Müntz polynomials, which can be regarded as continuation of our previous work. The extension is twofold. Firstly, the existing Müntz spectral method for fractional differential equation with fractional derivative order $0<\mu<1$ is generalized to $0<\mu\le 2$, which is nontrivial since the classical Müntz polynomials only have no more than $H^{1/2+\lambda}$ regularity, where $0<\lambda\le 1$ is the characteristic parameter of the Müntz polynomial. Secondly, 1D Müntz spectral method is extended to the 2D space-fractional convection-diffusion equation. Compared to the time-fractional diffusion equation, some new operators such as suitable H^1-projectors are needed to analyze the error of the numerical solution. The main contribution of the present paper consists of an efficient method combining the Crank-Nicolson scheme for the temporal discretization and a new spectral method using the Müntz Jacobi polynomials for the spatial discretization of the 2D space-fractional convection-diffusion equation. A detailed convergence analysis is carried out, and several error estimates are established. Finally a series of numerical experiments is performed to verify the theoretical claims.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Two-dimension space-fractional convection-diffusion equation, singularity, Müntz polynomials, Müntz spectral method, Crank-Nicolson scheme, spectral accuracy.

1 Introduction

In the past two decades, fractional differential equations (FDEs) have gained increasing popularity, mainly due to its concrete applications in variety of fields such as control the-

*Corresponding author. *Email addresses:* `cjxu@xmu.edu.cn` (C. Xu), `dmhou@stu.xmu.edu.cn` (D. Hou), `azaiez@enscbp.fr` (M. Azaiez)

ory, biology, electro-chemical processes, viscoelastic materials, polymers, finance, etc. On the other side, interesting features of FDEs have also been attracting much attention from theoretical and numerical researchers. There exists an enormous amount of literature on numerical methods for FDEs and it is impossible to give a complete list of references. Among the existing methods, we mention [1, 10–12, 22, 25, 26, 35, 47] for various time discretizations for the time-fractional diffusion equation. Concerning space-fractional differential equations, we refer to [6, 8, 24, 29, 30, 37–40, 42, 49] and the references therein for a variety of developed numerical methods, covering from finite difference methods to finite element methods.

It has become well-known now that solving fractional differential equations is accompanying a number of features/difficulties, including the non-locality of fractional differential operators, singularity/low regularity of the kernel functions and the solutions of the associated problems. The first feature unavoidably results in high storage cost for traditional approaches. This difficulty has led to the investigation of fast evaluation techniques using the approximation to the weakly singular kernel function by a sum-of-exponentials, resulting in some stepping schemes with reduced storage; see, e.g., Baffet and Hesthaven [2, 3], Jiang et al. [17, 48], Yan et al. [41], Zeng et al. [46], etc.

High order methods have also been considered as an efficient way to reduce the memory requirement. This consideration has motivated several researches, e.g., [13, 15, 16, 18–20, 27, 28, 33, 34, 45], in which various spectral methods have been constructed for time-fractional, space-fractional, or time-space-fractional differential equations. These methods could be highly efficient if the exact solution is smooth since the exponential convergence can be expected in this case. However, as it is aforementioned, the solution of FDEs is usually non-smooth due to the presence of the weakly singular kernel function in the definitions of fractional operators. Several other work have targeted at spectral methods for non smooth solutions. Among these methods, Zheng et al. [50] proposed a spectral method for the multi-term time-fractional diffusion equation. Zayernouri et al. [43] considered a Petrov-Galerkin spectral method using the so-called polyfractonomials, introduced in Zayernouri and Karniadakis [44]. Chen et al. [5] used generalized Jacobi functions to construct Petrov-Galerkin methods for a class of fractional initial/boundary value problems. In particular, Hou et al. [13, 15] introduced a general framework using Müntz polynomials for some weakly singular integro-differential equations and fractional differential equations. Both numerical experiments and theoretical analysis have shown that the computed solution by the Müntz spectral method can be exponentially accurate for a large class of fractional differential equations, even if the exact solution of these equations is not smooth.

The present work can be regarded as continuation of [13, 15]. This extension is motivated by the fact that the singularity caused by the 2D domain corners has form similar to $(x^2+y^2)^\rho$, x^ρ, y^ρ, or their combination for some real number ρ, which could be efficiently approximated by Müntz polynomials. Note that the order of the fractional derivatives in our previous work [13, 15] is not more than 1, thus the Müntz spectral method can be easily applied for those problems for the reason that the classical Müntz polynomials