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Convergence Analysis of Exponential Time Differencing Schemes for the Cahn-Hilliard Equation†

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Abstract. In this paper, we rigorously prove the convergence of fully discrete first- and second-order exponential time differencing schemes for solving the Cahn-Hilliard equation. Our analyses mainly follow the standard procedure with the consistency and stability estimates for numerical error functions, while the technique of higher-order consistency analysis is adopted in order to obtain the uniform L^∞ boundedness of the numerical solutions under some moderate constraints on the time step and spatial mesh sizes. This paper provides a theoretical support for numerical analysis of exponential time differencing and other related numerical methods for phase field models, in which an assumption on the uniform L^∞ boundedness is usually needed.

AMS subject classifications: 35K55, 65M12, 65M15, 65F30

Key words: Cahn-Hilliard equation, exponential time differencing, convergence analysis, uniform L^∞ boundedness.

1 Introduction

In this paper, we consider the Cahn-Hilliard equation [5],

$$u_t = -\varepsilon^2 \Delta^2 u + \Delta f(u), \quad \boldsymbol{x} \in \Omega,\ t \in (0,T] \tag{1.1}$$

with $f(u) = u^3 - u$, where Ω is a rectangle in $\mathbb{R}^2$ or a cuboid in $\mathbb{R}^3$ and $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is the unknown function subject to the periodic boundary condition. An important feature of the Cahn-Hilliard equation (1.1) is that it can be regarded as the H^{-1} gradient flow with respect to the Ginzburg-Landau energy functional

$$E(u) = \int_\Omega \Big(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \Big) \mathrm{d}\boldsymbol{x} \tag{1.2}$$

†This paper is dedicated to Professor Jie Shen on the occasion of his 60th birthday.

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with $F(u)=\frac{1}{4}(u^2-1)^2$, and thus the solution of (1.1) satisfies the energy law (noting that $f(u)=F'(u)$):

$$\frac{\mathrm{d}E(u)}{\mathrm{d}t}=-\int_\Omega |\nabla(-\varepsilon^2\Delta u+f(u))|^2\mathrm{d}\boldsymbol{x}\leq 0, \tag{1.3}$$

i.e., the energy E is decreasing along the time. As one of the typical systems for phase field modeling, the Cahn-Hilliard equation has been widely used to model the phase separations and accumulation occurring in mixtures of small molecules and some other moving interface problems involving mass-conserved order parameters (see, e.g., [2,3,6, 10,33]). Thus, accurate and stable temporal discretizations for the Cahn-Hilliard equation are important for large scale and long-time simulations of coarsening dynamics.

In recent years, numerous numerical methods have been proposed for solving the Cahn-Hilliard equation and many other phase field models, in which the discrete version of the energy law (1.3) attracts much attention in numerical analysis. The modified Crank-Nicolson scheme [11,17,41] and the convex splitting scheme [14] are proven to be unconditionally energy stable (see also [34,35,39]). These schemes are usually nonlinear so that they are time-consuming due to the need of nonlinear solvers in each time step.

To avoid the nonlinear iterations, a first-order linear scheme was constructed in [21] by adding an extra stabilization term to the classic semi-implicit scheme for the Cahn-Hilliard equation. This stabilized scheme is unconditionally energy stable if the stabilizing parameter satisfies some certain inequality which depends on the L^∞ bound of the numerical solutions. Such scheme has been also applied to some other phase field models [8,40] with the same assumptions needed. Later, the first- and second-order stabilized schemes were studied more systematically in [16,37] under an assumption on the Lipschitz continuity of the nonlinear term $f(u)$. Since $f(u)$ is a polynomial of degree three in (1.1), the assumption on the Lipschitz continuity of $f(u)$ is in fact equivalent to the uniform L^∞ boundedness of the numerical solutions. Recently, an exponential time differencing (ETD) method for the Cahn-Hilliard equation was proposed in [28] based on the same stabilizing technique, and the stabilizing parameter is also required to depend on the numerical solutions to guarantee the energy stability. In addition, the classic backward Euler scheme was analyzed in [13] and the error estimate was also derived by assuming the uniform L^∞ boundedness of the numerical solutions.

Since the energy law (1.3) in the PDE level holds without any extra requirements, it is highly desired to remove the L^∞ assumption on the numerical solutions for theoretical completeness of stability and convergence analysis. By using advanced harmonic analysis for the stabilized scheme, the authors of [32] removed the technical restrictions and established the unconditional energy stability of the stabilized scheme for general phase field models. The energy stability of the second-order stabilized scheme in 2D and 3D spaces was analyzed in [30, 31] by using the similar approach. On the other hand, some investigations were devoted to the theoretical justification of the uniform L^∞ boundedness. It was shown in [4] that for a truncated potential $F(u)$ with quadratic growth at infinities, the maximum norm of the solution of the Cahn-Hilliard equation is bounded. The technique with a truncated potential was also considered in the literature,