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A Fast State-Based Peridynamic Numerical Model

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Abstract. The peridynamic (PD) theory is a reformulation of the classical theory of continuum solid mechanics and is particularly suitable for the representation of discontinuities in displacement fields and the description of cracks and their evolution in materials, which the classical partial differential equation (PDE) models tend to fail to apply. However, the PD models yield numerical methods with dense stiffness matrices which requires $\mathcal{O}(N^2)$ memory and $\mathcal{O}(N^3)$ computational complexity where N is the number of spatial unknowns. Consequently, the PD models are deemed to be computationally very expensive especially for problems in multiple space dimensions.

State-based PD models, which were developed lately, can be treated as a great improvement of the previous bond-based PD models. The state-based PD models have more complicated structures than the bond-based PD models. In this paper we develop a fast collocation method for a state-based linear PD model by exploring the structure of the stiffness matrix of the numerical method. The method has an $\mathcal{O}(N)$ memory requirement and computational complexity of $\mathcal{O}(N\log N)$ per Krylov subspace iteration. Numerical methods are presented to show the utility of the method.

AMS subject classifications: 65F10, 65M70, 65T50, 74B15

Key words: State-based peridynamic model, fast algorithm, collocation method, stiffness matrix.

1 Introduction

The classical theory of continuum solid mechanics assumes that all internal forces act locally and yields mathematical models that are expressed in terms of partial differential equations (PDEs). While they have been very successful for problems with smooth displacement fields, these models tend to have difficulties to describe problems with evolving discontinuities, due to their differentiability assumption on displacement fields. The

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peridynamic (PD) theory [17] is a reformulation of the classical theory of continuum solid mechanics, and yields nonlocal mathematical formulations that are based on long-range interactions. Constitutive models in the PD theory depend on finite deformation vectors, instead of deformation gradients in classical constitutive models. Consequently, PD models are particularly suitable for the representation of discontinuities in displacement fields and the description of cracks and their evolution in materials. They have been successfully applied to applications such as failure and damage in composite laminates [13], crack propagation and branching [10], crack nucleation [20], phase transformations in solids [4], impact damage [16,18], polycrystal fracture [9], crystal plasticity [21], damage in concrete [8], and geomaterial fragmentation [11].

Because the PD models provide better modeling capability than classical integer-order PDE models do especially for problems with discontinuous displacement fields, different numerical methods have been developed for these models with different computational expenses, memory requirements, and implementation effort as well as accuracy, convergence, and stability [6]. For instance, collocation methods and meshfree methods apply directly to the strong form of the PD models and are relatively simple to implement [14,15,18], while Galerkin finite element methods apply to the weak form of the PD models and enjoy high convergence rates [2,13,23,24]. It is worth mentioning that in [24] the authors established a framework to develop robust asymptotically compatible schemes for nonlocal models and their local limits.

However, the PD models present numerical difficulties that were not encountered in classical integer-order PDE models. Recall that the numerical methods for integer-order PDE models generate sparse stiffness matrices which have an optimal-order memory requirement of $\mathcal{O}(N)$ where N is the number of spatial unknowns. The matrix-vector multiplication has a computational complexity of $\mathcal{O}(N)$. Hence, the computational complexity of any Krylov subspace iterative solver depends only on the number of iterations. In contrast, the numerical methods for the PD models yield dense stiffness matrices. Consequently, the numerical methods for PD models have $\mathcal{O}(N^2)$ memory requirement. Direct solvers have been widely used in the numerical simulation of PD models, which have $\mathcal{O}(N^3)$ computational complexity to invert the coefficient matrices. On the other hand, the matrix-vector multiplication by the stiffness matrices have $\mathcal{O}(N^2)$ computational complexity. Furthermore, the evaluation and assembly of the stiffness matrices require the evaluation of $\mathcal{O}(N^2)$ entries, which can be very expensive and often constitute a very large portion of simulation times! In summary, the significantly increased computational complexity and memory requirement of the PD models over those for the classical integer-order PDE models render the numerical simulation of the PD models computationally very expensive, especially for problems in multiple space dimensions.

We previously developed fast and accurate numerical methods for bond-based linear PD models in one and two space dimensions. These fast methods reduce the memory requirement from $\mathcal{O}(N^2)$ to the optimal-order memory requirement of $\mathcal{O}(N)$ and computational complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(N\log N)$ per Krylov subspace iteration, which were achieved without resorting to any lossy compression, but rather by exploring the