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# Lévy Backward SDE Filter for Jump Diffusion Processes and Its Applications in Material Sciences

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**Abstract.** We develop a novel numerical method for solving the nonlinear filtering problem of jump diffusion processes. The methodology is based on numerical approximation of backward stochastic differential equation systems driven by jump diffusion processes and we apply adaptive meshfree approximation to improve the efficiency of numerical algorithms. We then use the developed method to solve atom tracking problems in material science applications. Numerical experiments are carried out for both classic nonlinear filtering of jump diffusion processes and the application of nonlinear filtering problems in tracking atoms in material science problems.

**AMS subject classifications**: 65C30, 65K10

**Key words**: Nonlinear filtering problem, backward SDEs, jump diffusion processes, material sciences.

## 1 Introduction

The nonlinear filtering problem is one of the key missions in data assimilation, in which observations of a system are incorporated into the state of a numerical model of that system. Mathematically, the nonlinear filtering problem is to obtain, recursively in time, the best estimate of the state of unobservable stochastic dynamics $S=\{S_t : t \geq 0\}$, based on an associated observation process, $M=\{M_t : t \geq 0\}$, whose values are a function of $S$ after corruption by noises. This suggests the optimal filtering problem of obtaining the conditional distribution of the state $S_t$ from the observations up until time $t$, which

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achieves the best estimate of this distribution, in the squared error sense, based on the available observations.

The nonlinear filtering theory finds its applications in numerous scientific and engineering research areas, such as target tracking [17, 34], signal processing [26, 38], image processing [51, 52], biology [16, 37, 53], or mathematical finance [10, 19, 21]. Some of the pioneer contributions to the development of nonlinear filters are due to Kushner [36] and Stratonovich [54]. Later, Zakai [59] introduced an alternative approach to the computation of the nonlinear filter by developing the so-called Zakai equation, which is a stochastic partial differential equation (SPDE), and the best estimate of the nonlinear filter, i.e. the conditional distribution, is represented by the solution of the Zakai equation. Although the Zakai's approach produces the "*exact*" solution of the nonlinear filtering problem in theory, solving the SPDEs numerically can be extraordinarily difficult, especially when the state processes are in high dimensions [5, 23, 32, 60]. A more widely accepted method by practitioners to solve the nonlinear filtering problem is the sequential Monte Carlo approach, which is also known as the particle filter method [4, 11, 13, 18, 24, 33, 40, 41]. The particle filter method uses a number of independent random variables, called particles, sampled directly from the state space to represent the prior probability, and updates the prior by including the new observation to get the posterior. This particle system is properly located, weighted and propagated recursively according to Bayes' theorem. As a Monte Carlo approach, with sufficient large number of samples the particle filter provides an accurate representation of the state probability density function (pdf) as desired in the nonlinear filtering problem. Convergence of a particle filter to the optimal filter was shown under certain conditions [14, 15, 27]. In addition to the Zakai's approach and Monte Carlo type approach, the authors have developed an alternative method, which solves the nonlinear filtering problem through a forward backward doubly stochastic differential equations (BDSDEs) system. The theoretical basis of the BDSDEs approach is the fact that the BDSDEs system is equivalent to a parabolic type SPDE and the solution of that system is the conditional distribution of the state as required in the nonlinear filtering problem [3, 6–8]. In this connection, it produces the exact solution of the nonlinear filtering problem, just like the Zakai's approach. In the meantime, as a stochastic *ordinary* differential equation (SDE) approach, it also relies on stochastic sampling, just like the particle filter method. Therefore, the BDSDEs approach builds the bridge between the Zakai's approach and the Monte Carlo type approach.

In this paper, we consider a more general nonlinear filtering problem – the nonlinear filtering problem for jump diffusion processes, in which the state process $S_t$ is a jump diffusion process and the state dynamic is perturbed by both traditional Gaussian noises and other kinds of Lévy type noises. Different from classical nonlinear filtering problems, numerical methods to solve the nonlinear filtering problem for jump diffusion processes are not well developed. The existing methods for solving this type of problems focus on numerical approximation for its corresponding Zakai equation [43, 49, 50]. However, due to the nonlocal behavior of the state dynamics as a jump diffusion process, the corresponding Zakai equation contains fractional derivatives in spatial dimension, which is a