Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0047

Vol. **27**, No. 3, pp. 639-671
March 2020

Entropies and Symmetrization of Hyperbolic Stochastic Galerkin Formulations

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Received 25 March 2019; Accepted (in revised version) 21 August 2019

Abstract. Stochastic quantities of interest are expanded in generalized polynomial chaos expansions using stochastic Galerkin methods. An application to hyperbolic differential equations does in general not transfer hyperbolicity to the coefficients of the truncated series expansion. For the Haar basis and for piecewise linear multiwavelets we present convex entropies for the systems of coefficients of the one-dimensional shallow water equations by using the Roe variable transform. This allows to obtain hyperbolicity, wellposedness and energy estimates.

AMS subject classifications: 35L65, 54C70, 58J45, 70S10, 37L45, 35R60

Key words: Hyperbolic partial differential equations, uncertainty quantification, stochastic Galerkin, shallow water equations, wellposedness, entropy, Roe variable transform.

1 Introduction

Wellposedness is an important property that systems of partial differential equations (PDEs) should fulfil. Wellposedness means the solution exists, it is unique and the solution depends continuously on initial conditions [35]. Classical solutions to most hyperbolic conservation laws have this property, which explains, why these equations are widely used to model fluid dynamics [49] and other applications like traffic flow [50]. Most physically motivated systems are endowed with an entropy that describes the decay of energy, which in turn guarantees well-posed classical solutions [9,29,45]. A famous example is the physical entropy for Euler and shallow water equations, see e.g. [18].

Classical solutions, however, exist in finite time only up to the possible occurrence of shocks [65]. Therefore, weak solutions are considered which are not necessarily unique. Existence and uniqueness of bounded weak entropy solutions have been shown in [45]

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using entropy-entropy flux pairs. All of these entropy-entropy flux pairs must satisfy an entropy inequality. In the *scalar* case a strictly convex flux function and *one* entropy-entropy flux pair is sufficient to characterize the entropy solution uniquely [47, 59]. This result could not been extended to arbitrary systems, when entropies rarely exist or remain unknown [47]. A single entropy-entropy flux pair, however, manages to weed out all but one weak solution, as long as a classical solution exists [18]. Thus, an entropy transfers the wellposedness of classical solutions to a weak formulation.

When initial data are not known exactly, but are given by their probability law or by statistical moments, the deterministic entropy concepts should be extended to the stochastic case. A mathematical framework for random entropy solutions of scalar random hyperbolic equations is developed in [55, 75]. It is shown that existing statistical moments in the initial conditions are transferred to the solution. In this non-intrusive point of view, first *pointwise* entropy solutions are determined, then the statistics of interest are computed. If there is only interest in the statistics, non-intrusive methods have been proven successful in previous works [1, 3, 13, 17, 56, 66, 68, 73, 74, 78] and are often preferred in practice, since deterministic solvers can be used. In particular for shallow water equations, results are available in several spatial and random dimensions [57].

Desirable deterministic schemes are in smooth regions high-order accurate, but can also resolve singularities in an essentially nonoscillatory (ENO) fashion. WENO schemes consist of a weighted combination of local reconstructions on different stencils. Some schemes allow unstructured grid in higher dimensions [37, 69]. In particular for balance laws, centered CWENO schemes [15, 16, 43] can reconstruct also the source term.

In contrast to non-intrusive methods, we investigate the intrusive stochastic Galerkin method. Stochastic processes are represented as orthogonal functions, for instance orthogonal polynomials and multiwavelets. These representations are known as generalized polynomial chaos (gPC) expansions [7, 25, 30, 79, 82]. Expansions of the stochastic input are substituted into the governing equations and they are projected to obtain deterministic evolution equations for the gPC coefficients. The applications of this procedure have been proven successful for diffusion [22, 83] and kinetic equations [8, 38, 42, 70, 85]. In general, results for hyperbolic systems are not available [20, 21, 53], since desired properties like hyperbolicity and the existence of entropies are not transferred to the intrusive formulation. A problem is posed by the fact that the deterministic Jacobian of the projected system differs from the random Jacobian of the original system and therefore not even real eigenvalues, which are necessary for hyperbolicity, are guaranteed in general.

In particular for a stochastic Galerkin formulation of shallow water equations, the loss of hyperbolicity and hence the loss of all entropy-entropy flux pairs is proven in [21, Prop. 2]. Also stochastic Galerkin formulations for isothermal Euler equations are in general not hyperbolic [24, 40].

So far, a serious problem for both non- and intrusive methods remains the convergence in the stochastic space. Methods are desirable that allow estimates and convergence results for a smooth dependency on the stochastic input. Convergence results in previous works are based on smoothness assumptions, although solutions to hyperbolic