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Construction of the Local Structure-Preserving Algorithms for the General Multi-Symplectic Hamiltonian System

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Abstract. Many partial differential equations can be written as a multi-symplectic Hamiltonian system, which has three local conservation laws, namely multi-symplectic conservation law, local energy conservation law and local momentum conservation law. In this paper, we systematically give a unified framework to construct the local structure-preserving algorithms for general conservative partial differential equations starting from the multi-symplectic formulation and using the concatenating method. We construct four multi-symplectic algorithms, two local energy-preserving algorithms and two local momentum-preserving algorithms, which are independent of the boundary conditions and can be used to integrate any partial differential equations written in multi-symplectic Hamiltonian form. Among these algorithms, some have been discussed and widely used before while most are novel schemes. These algorithms are illustrated by the nonlinear Schrödinger equation and the Klein-Gordon-Schrödinger equation. Numerical experiments are conducted to show the good performance of the proposed methods.

AMS subject classifications: 65L12, 65M06, 65M12

Key words: Multi-symplectic formulation, multi-symplectic algorithm, energy-preserving algorithm, momentum-preserving algorithm, concatenating method, average vector field method.

1 Introduction

Recently, there has been an increased emphasis on constructing numerical algorithms to preserve the intrinsic properties of the original problems in the continuous dynamical

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systems. The earliest attempt can go back to 1928 when Courant, Friedrichs and Lewy constructed a 5-point finite difference scheme which yields a global invariant form [1]. Methods that can conserve at least some of the structural properties of systems are called geometric integrators or structure-preserving algorithms. Lots of researchers have obtained a series of important results on construction and theory analysis of structure-preserving algorithms for Hamiltonian ordinary differential equations (ODEs) [2–4]. It is known that Hamiltonian partial differential equations (PDEs) arise as models in meteorology and weather prediction, nonlinear optics, solid mechanics and electromagnetism, cosmology and quantum field theory, and so on. As geometric integration has gained remarkable success in the numerical analysis of ODEs, it is desirable to extend the idea of geometric integration to solve PDEs. A popular method to treat Hamiltonian PDEs is the so-called method of lines in which a PDE is first discretized in space direction resulting in a large system of Hamiltonian ODEs. Then the resulted ODEs are integrated by a structure-preserving algorithm. However, no method is mature to guarantee the resulted ODEs to be Hamiltonian.

The introduction of multi-symplectic formulation [5, 6] provides a new way to solve the conservative PDEs based on multi-symplectic geometric integrators. Numerous conservative PDEs, such as the nonlinear Schrödinger (NLS) equation, the Klein-Gordon-Schrödinger (KGS) equation, the Korteweg-de Vries (KdV) equation, the Camassa-Holm (CH) equation, the Maxwell equation, the Landau-Lifshitz (LL) equation and so on can be written in this form, from which the intrinsic multi-symplectic conservation law, local energy and momentum conservation laws can be naturally derived. Compared with the classical Hamiltonian form, multi-symplectic Hamiltonian form considers space and time on an equal footing and is well suited for numerical discretization methods that emphasize local properties. Afterwards, to preserve the multi-symplectic structure, multi-symplectic algorithms developed very fast and a lot of important achievements have been obtained in past decades. For more details, please refer to review articles [7] and references therein.

We all know that the physical conservation law in regard of system energy or momentum plays a significant role in the study of properties of solutions, especially in the theory of solitons. Extensive numerical studies have been developed for the conservation of those physical quantities [8–19]. Note that these two physical invariants are local invariants, which exist in any time-space region exactly independent of boundary conditions. Cai et al. [20] proposed some local energy-preserving for the coupled nonlinear Schrödinger equation in 2013. Wang et al. [21] also gave several local energy-preserving schemes for KdV equation. However, these methods in References are not studied systematically either in their presentations or in their applications, that is to say, they are investigated for some particular equation. The methods discussed here are, by contrast, presented systematically and in a unified framework which can be applied to a large class of Hamiltonian PDEs. In other words, we start from the multi-symplectic formulation and construct a series local structure-preserving algorithms (LSPAs), which can be refined as a unified framework. Then, we can use this framework to construct LSPAs