

A New Type of High-Order WENO Schemes for Hamilton-Jacobi Equations on Triangular Meshes

Jun Zhu¹ and Jianxian Qiu^{2,*}

¹ College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, P.R. China.

² School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High-Performance Scientific Computation, Xiamen University, Xiamen, Fujian 361005, P.R. China.

Received 10 June 2018; Accepted (in revised version) 18 October 2018

Abstract. In this paper, a new type of third-order and fourth-order weighted essentially non-oscillatory (WENO) schemes is designed for simulating the Hamilton-Jacobi equations on triangular meshes. We design such schemes with the use of the nodal information defined on five unequal-sized spatial stencils, the application of monotone Hamiltonians as a building block, the artificial set of positive linear weights to make up high-order approximations in smooth regions simultaneously avoiding spurious oscillations nearby discontinuities of the derivatives of the solutions. The spatial reconstructions are convex combinations of the derivatives of a modified cubic/quartic polynomial defined on a big spatial stencil and four quadratic polynomials defined on small spatial stencils, and a third-order TVD Runge-Kutta method is used for the time discretization. The main advantages of these WENO schemes are their efficiency, simplicity, and can be easily implemented to higher dimensional unstructured meshes. Extensive numerical tests are performed to illustrate the good performance of such new WENO schemes.

AMS subject classifications: 65M60, 35L65

Key words: Unequal-sized stencil, weighted essentially non-oscillatory scheme, high-order approximation, Hamilton-Jacobi equation, triangular mesh.

1 Introduction

In this paper, we design a class of new third-order and fourth-order weighted essentially non-oscillatory (WENO) schemes for solving the Hamilton-Jacobi equations

$$\begin{cases} \phi_t + H(x, y, t, \phi, \phi_x, \phi_y) = 0, \\ \phi(x, y, 0) = \phi_0(x, y), \end{cases} \quad (1.1)$$

*Corresponding author. *Email addresses:* zhu jun@nuaa.edu.cn (J. Zhu), jxqiu@xmu.edu.cn (J. Qiu)

on triangular meshes. It is well known that the Hamilton-Jacobi (HJ) equations are often used in the applications of differential games, geometric optics, computer vision, variational calculus, control theory, etching, robotic navigation, and crystal growth [11, 33, 43]. The solution of (1.1) is continuous but the derivatives of the solution may have discontinuities or generate singularities via time approaching.

The concepts of the entropy conditions and the definition of the viscosity solution were formulated in [13–15, 47]. Abgrall and Sonar [2] pointed out that the viscosity solutions of the HJ equations may not be unique with the consideration of the physical implications. The HJ equations show very close relationship to the conservation laws and the numerical methods for HJ equation are similar to those for the conservation laws [4, 39, 40]. In 1984, Crandall and Lions [16] proposed first-order monotone finite difference schemes and then indicated that such schemes could converge to the viscosity solution of (1.1). Osher and Sethian [35] proposed a second-order essentially non-oscillatory (ENO) scheme for solving the HJ equations. Osher and Shu [36] designed high-order accurate ENO schemes for solving the HJ equations. Lafon and Osher [25] proposed unstructured ENO schemes for solving the HJ equations. In 2000, Jiang and Peng [21] proposed finite difference high-order weighted ENO (WENO) [22, 31, 34] scheme for solving the HJ equations on structured meshes which used the similar framework proposed by Jiang and Shu [22] for the conservation laws. Li and Chan [30], and Zhang and Shu [49] also proposed unstructured different finite difference high-order WENO schemes for solving the HJ equations in two dimensions. Herein, Qiu [37, 38], and Qiu and Shu [41] designed Hermite WENO (HWENO) schemes based on the finite volume and finite difference frameworks for solving the HJ equations on structured meshes. The central high resolution schemes for the HJ equations were presented by a series of literature, e.g. [7–9, 24, 27, 32]. Some schemes, such as weighted power ENO schemes [42], mapped WENO schemes [9, 18], discontinuous Galerkin schemes [12] and relaxation schemes [23] *et al.*, were also used to solve for the HJ equations. In [3, 5, 19, 26], some finite element methods were constructed on unstructured meshes. Hu and Shu [19] proposed discontinuous Galerkin methods for solving the HJ equations. In 2011, Yan and Osher [48] gave a local discontinuous Galerkin method for solving the HJ equations.

This paper is a new extension of [54] from finite volume schemes for the conservation laws to finite difference schemes for the HJ equations, based on the similar spirit of WENO methodologies specified in [49]. The major advantage of such new WENO schemes is their easy implementation in the computation. These new WENO schemes have convex combinations of x - or y -directional derivatives of one modified high degree polynomial and four low degree polynomials. The essential merits of such methodology are its robustness in spatial field by the definition of any positive linear weights, and only one central big spatial stencil and four biased or central small stencils are used to reconstruct five different degree polynomials. Therefore, we apply the derivatives of the high degree polynomial defined on central big spatial stencil for obtaining high-order numerical approximations of $\nabla\phi$ at different vertexes in smooth regions and switch to the derivatives of quadratic degree polynomials defined on biased or central small spa-