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A New Interpolation for Auxiliary Unknowns of the Monotone Finite Volume Scheme for 3D Diffusion Equations

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Abstract. A monotone cell-centered finite volume scheme for diffusion equations on tetrahedral meshes is established in this paper, which deals with tensor diffusion coefficients and strong discontinuous diffusion coefficients. The first novelty here is to propose a new method of interpolating vertex unknowns (auxiliary unknowns) with cell-centered unknowns (primary unknowns), in which a sufficient condition is given to guarantee the non-negativity of vertex unknowns. The second novelty of this paper is to devise a modified Anderson acceleration, which is based on an iterative combination of vertex unknowns and will be denoted as AA-Vertex algorithm, in order to solve the nonlinear scheme efficiently. Numerical testes indicate that our new method can obtain almost second order accuracy and is more accurate than some existing methods. Furthermore, with the same accuracy, the modified Anderson acceleration is much more efficient than the usual one.

AMS subject classifications: 65M08, 65M12, 65B99

Key words: Diffusion equations, tetrahedral meshes, finite volume scheme, monotonicity, Anderson acceleration.

1 Introduction

For solving numerically diffusion equations, it not only requires the desired accuracy and stability, but also needs to satisfy the basic properties of physical model, such as conservation and monotonicity etc.. Generally, the monotonicity refers to be non-negativity

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preserving, which is a special case of the minimum principle. In the setting of linear diffusion problems the extremum property is equivalent with the monotonicity.

It is well known that some classical finite volume (FV) schemes [1, 2] or finite element (FE) schemes [3, 4] do not satisfy monotonicity unconditionally on meshes with large distortion or for diffusion problems with strongly anisotropic coefficients. Various techniques, such as repair technique [5–7], mesh optimization [8,9] and so on, have been proposed to reduce spurious oscillations. Huang and Wang [10] investigate the preservation of discrete maximum principle (DMP) by a weak Galerkin approximation of boundary values problem with a general anisotropic diffusion coefficient. The authors of [11] prove that when the discretization satisfies local conservation and linear preservation, it is impossible to establish a linear nine-point scheme unconditionally satisfying the monotonicity criterion. Then nonlinear monotone schemes have been introduced [12–16]. A nonlinear stabilized Galerkin approximation of the Laplace operator has been presented in [12]. A nonlinear monotone FV scheme with two-point flux structure is proposed in [13]. Based on the same idea, an interpolation-free monotone FV scheme is presented in [15]. An adaptive approach of constructing discrete flux to guarantee monotonicity of the FV scheme on any star-shaped polygonal meshes is proposed by Yuan et al. [16]. Its further development and application can be found in [17–20].

In the construction of some nonlinear schemes, in addition to primary unknowns, the auxiliary unknowns are also introduced to treat discontinuous coefficients rigorously. In order to get the scheme only with primary unknowns, the auxiliary unknowns need to be represented by primary unknowns. This procedure is important, since it affects the monotonicity and accuracy of the scheme. The scheme in [21] has only cell-centered unknowns, and the vertex unknowns are approximated by the arithmetic average of the neighboring cell-centered unknowns. The inverse distance weighted method is first introduced in [22]. A common defect of the two methods above is that there is remarkable accuracy loss for discontinuous coefficient problems or on distorted meshes. A method of harmonic averaging points is introduced in detail in [23], in which auxiliary unknowns are defined at the harmonic averaging points. The authors of [2] give a more accurate and complex method of interpolating vertex unknowns, which is used in another work [16]. In the case of diffusion coefficient being continuous, Taylor expansion is used to determine the weighted coefficients. In the case of diffusion coefficient being discontinuous, a method based on the continuity of normal flux components and tangential gradients is presented. However, the computational cost of such a method is too large for the three dimensional (3D) problem with discontinuous coefficients.

There are also some FV schemes on 3D polyhedral meshes, such as [24–30], which do not satisfy the monotonicity. A method [27] of interpolating vertex unknowns based on the linear weighted combination method similar to 2D nine-point FV scheme is proposed for 3D problems, however, the weighted coefficients cannot be guaranteed to be non-negative. The discrete duality FV method (DDFV) for 3D problems is proposed by Hermeline [29, 30]. In this method, the author integrates the diffusion equations over both a primal mesh and an associated dual mesh, hence both the cell-centered unknowns