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Vectorial Kinetic Relaxation Model with Central Velocity. Application to Implicit Relaxations Schemes

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Abstract. We apply flux vector splitting (FVS) strategy to the implicit kinetic schemes for hyperbolic systems. It enables to increase the accuracy of the method compared to classical kinetic schemes while still using large time steps compared to the characteristic speeds of the problem. The method also allows to tackle multi-scale problems, such as the low Mach number limit, for which wave speeds with large ratio are involved. We present several possible kinetic relaxation schemes based on FVS and compare them on one-dimensional test-cases. We discuss stability issues for this kind of method.

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1 Introduction

Hyperbolic systems often involve multi-scale phenomena, resulting from different characteristic speeds. A typical example is given by material and acoustic waves in the (compressible) Euler system. In order to avoid overly stringent stability condition due to the fast waves, implicit or semi-implicit methods have been developed, see for instance [1, 3, 4, 6, 18, 25, 33] and references therein. However, they often lead to large computational cost due to the matrix assembly, storage and inversion. As introduced in [16], an alternative is to consider implicit methods based on vectorial kinetic relaxation schemes, which decouple linear transport from the non-linear dynamics. However, they generally suffer from a loss of accuracy in space and/or in time when the numerical

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errors on the slow waves depends on the amplitude of the fast ones. In this paper, we study and compare several variants of this method, which are based on the Flux Vector Splitting strategy and show that they allow to better capture the various waves and keep a good accuracy.

Kinetic BGK (Bhatnagar, Gross and Krook) relaxation methods have been introduced to solve hyperbolic systems, see [7,30] and references therein. The unknown of the hyperbolic system is expressed as the macroscopic moment of a kinetic distribution function, whose components correspond to various velocities. The kinetic distribution function satisfies a transport equation at constant velocities with a BGK source term, that makes the kinetic distribution relax to a Maxwellian distribution depending only on the macroscopic quantities. Consistency relations link Maxwellian distribution and the fluxes so that the kinetic equation tends to the hyperbolic system in the limit of large relaxation frequency.

Vectorial kinetic BGK relaxation methods, introduced in [2,30], are a systematic way to devise relaxation methods for any hyperbolic system: each component of the hyperbolic system is represented with the same number of components in the kinetic vector, which are associated with the same set of velocities. This is the strategy we consider in this work. Note that the Jin-Xin relaxation method [26] enters in this class of methods. Other kinetic BGK relaxation methods take advantage of the structure of the hyperbolic problem to build less generic but more compact kinetic representations. They are intensively used in the lattice Boltzmann community, see for instance [13]. We also mention that several other relaxation systems have been developed for specific system, like for the Euler system. We refer to [9,11,14,31] and references therein. Up to our knowledge, equivalence with vectorial kinetic BGK relaxation methods has been pointed out only for Suliciu relaxation [8].

The kinetic BGK model has the advantage of concentrating the non-linearity of the system in the local source term: the transport term is linear with constant advection velocities. Therefore, implicit schemes can be easily implemented using splitting techniques. The transport equations can be solved independently using implicit schemes, but with an explicit cost, while the local source term is solved using for instance a θ-scheme. In [16], the transport equations were solved with a Discontinuous Galerkin solver. Here, we propose to use a high-order explicit semi-Lagrangian method on uniform meshes, that does not have stability constraints and thus enables to consider large time steps [17]. Moreover, the numerical resolution of the source term is rewritten as an algebraic relaxation step depending on a parameter $\omega \in]0,2]$: the relaxation step reduces to a projection to the equilibrium kinetic distribution for $\omega = 1$ and to a symmetry with respect to this equilibrium kinetic distribution when $\omega = 2$. For the latter case, the overall transport-relaxation leads to second-order time accuracy on the macroscopic hyperbolic unknown as pointed out in [16,19,20].

The simplest vectorial kinetic BGK model is the so-called $[D1Q2]^N$, where N is the size of the hyperbolic system and $[D1Q2]$ stands for dimension 1 with two velocities. It is equivalent to the Jin-Xin relaxation system [26]. The associated scheme has been already