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Simulation of Three-Dimensional Free-Surface Flows Using Two-Dimensional Multilayer Shallow Water Equations

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Abstract. We present an efficient and conservative Eulerian-Lagrangian method for solving two-dimensional hydrostatic multilayer shallow water flows with mass exchange between the vertical layers. The method consists of a projection finite volume method for the Eulerian stage and a method of characteristics to approximate the numerical fluxes for the Lagrangian stage. The proposed method is simple to implement, satisfies the conservation property and it can be used for multilayer shallow water equations on non-flat bathymetry including eddy viscosity and Coriolis forces. It offers a novel method of calculating stratified vertical velocities without the use of the Navier-Stokes equations. Numerical results are presented for several examples and the obtained results for a free-surface flow problem are in close agreement with the analytical solutions. We also test the performance of the proposed method for a test example of wind-driven flows with recirculation.

AMS subject classifications: 65M08, 35L53, 76B15, 74J40, 76B07

Key words: Multilayer shallow water equations, incompressible hydrostatic flows, Eulerian-Lagrangian scheme, finite volume solver, projection method.

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1 Introduction

Incompressible Navier-Stokes equations have been widely used in the literature to simulate water flows including eddy diffusion and Coriolis forces, see for example [13,26,31]. Further references on a general overview of shallow water wave modelling include [18–

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21] among others. However, for free-surface flows these models frequently become complicated due to the presence of moving boundaries within the flow domain and also due to the inclusion of hydrostatic pressure. Under certain assumptions these models can be replaced by the well-established shallow water equations. Indeed, the shallow water equations can be derived by depth-averaging the three-dimensional Navier-Stokes equations assuming that the pressure is hydrostatic and the vertical scale is far smaller than the horizontal scale, see [1] among others. In their depth-averaged form, shallow water equations have been used to model many engineering problems in hydraulics and free-surface flows including tides in coastal regions, rivers, reservoir and open channel flows among others, see for instance [12, 14, 23]. However, since these models are depth averaged, the vertical distribution of velocity field is not resolved and the bed friction is expressed only in terms of the mean velocity rather than the velocity near the bottom. Hence, the three-dimensional modeling of the hydrodynamic equations is needed for a better representation of the flow features, especially for recirculation flows and for solution of near-field problems involving sediment transport and thermal discharges in rivers and coastal waters.

Since standard shallow water models have been well developed, attention has been shifted to the shortcomings of this type of modeling namely, the use of single velocity profile for the entire depth of the fluid. This has been overcame with the recent introduction of multilayer shallow water equations for geophysical flows. Two-layer shallow water equations have been used to model immiscible fluids, see for example [2, 11, 17]. Multilayer shallow water equations with exchange terms have also been investigated in [3–5, 15] among others. These multilayer models can be derived by using a semi-discretization in the vertical direction of P_0 finite element types for the water velocity in the three-dimensional Navier-Stokes equations. The attractive points of this class of multilayer models include the fact that they avoid the computationally demanding methods required to solve the three-dimensional Navier-Stokes equations and at the same time providing stratified flow velocities as the pressure distribution is hydrostatic. Here, the flow problem can be approximated as a layered system made of multiple shallow fluids of distinct heights but with exchange terms between these layers. These fluids can also differ in terms of density, compressibility, viscosity and potential for mixing among others. During the last decades, multilayer shallow water models have attracted more attention and have been used for numerical simulation of a variety of hydrodynamical flows such as estuaries, bays and other nearshore regions where water flows interact with the bed geometry and wind shear stresses. However, most of these models consider only the one-dimensional version of the multilayer shallow water equations. To the best of our knowledge, simulation of two-dimensional multilayer shallow water equations with exchange terms is presented for the first time.

Developing highly accurate numerical solvers for multilayer shallow water equations presents a challenge due to the nonlinear aspect of these equations and their coupling through the source terms. More precisely, the difficulty in these multilayer models lies from the coupling terms involving some derivatives of the unknown physical variables