Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0029

Vol. **27**, No. 5, pp. 1470-1484
May 2020

Implementation of Finite Difference Weighted Compact Nonlinear Schemes with the Two-Stage Fourth-Order Accurate Temporal Discretization

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Received 23 February 2019; Accepted (in revised version) 1 September 2019

Abstract. In this paper, we present a new two-stage fourth-order finite difference weighted compact nonlinear scheme (WCNS) for hyperbolic conservation laws with special application to compressible Euler equations. To construct this algorithm, apart from the traditional WCNS for the spatial derivative, it was necessary to first construct a linear compact/explicit scheme utilizing time derivative of flux at midpoints, which, in turn, was solved by a generalized Riemann solver. Combining these two schemes, the fourth-order time accuracy was achieved using only the two-stage time-stepping technique. The final algorithm was numerically tested for various one-dimensional and two-dimensional cases. The results demonstrated that the proposed algorithm had an essentially similar performance as that based on the fourth-order Runge-Kutta method, while it required 25 percent less computational cost for one-dimensional cases, which is expected to decline further for multidimensional cases.

AMS subject classifications: 65M06, 65M20, 35L65, 35L04

Key words: Hyperbolic conservation laws, finite difference method, Lax-Wendroff type time discretization, WCNS.

1 Introduction

For a long time, the development of high-order numerical methods for hyperbolic conservation laws has been actively discussed. Among various high-order methods, the finite difference method (FDM), which can be constructed in a dimension-by-dimension manner for uniform/curvilinear grid, has been appreciated for its simplicity, effectiveness, and low computational cost [10].

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Considering FDM, many specific finite difference schemes have been proposed. It is worth mentioning examples such as the finite difference center/upwind compact scheme [9, 12], the finite-difference version of the weighted ENO (WENO) scheme [11, 19], and the weighted compact nonlinear scheme (WCNS) [7].

Recently, the WCNS is becoming more widely used as it has been proven to have important advantages, such as preserving freestream [5, 14], and satisfying geometric conservation law [6]. Typically, the WCNS is used jointly with the method of lines, which allows separating the spatial discretization from the time evolution. The spatial discretization of the WCNS includes the following steps: (1) nonlinear reconstruction of an arbitrary variable from a cell node to a cell edge, (2) evaluation of the numerical flux on a cell edge, and (3) calculation of high-order central finite difference of the numerical flux from a cell edge to a cell node. Once this discretization has been parsed, one may apply an appropriate ordinary differential equation solver to achieve higher order temporal accuracy. In this study we applied the high-order total variation diminishing (TVD) Runge-Kutta schemes for this purpose.

However, similar to most of methods using the Runge-Kutta time-stepping approach for time integration, generally, the WCNS will require using effective interpolation stencils. Furthermore, as noted in [17], there is an order barrier for TVD Runge-Kutta methods with positive coefficients: they cannot be higher than the fourth-order accuracy.

In fact, it is reasonable to construct multistage multiderivative algorithms for time integration [3]. Recently, the two-stage fourth-order time-accurate Lax-Wendroff (L-W) time solver was introduced, particularly for application with the hyperbolic conservation laws [13]. The two-stage L-W type time-stepping method is able to achieve fourth-order time accuracy solely owing to the use of both flux derivative and its time derivative. Previous works [4,8,15,16] have already demonstrated that this solver has advantages in terms of its computational efficiency.

Therefore, it is beneficial to investigate how to implement this two-stage fourth-order Lax-Wendroff type time solver in the WCNS. To the author's knowledge, there has been no previous research on this topic. In this work, we developed a new two-stage fourth-order WCNS with application of compressible Euler equations.

The rest of the paper is organized as follows. In Section 2, we provide a brief review of the two-stage fourth-order Lax-Wendroff time discretization method. In Section 3, we describe in detail the specific algorithms of the proposed approach. Extension to multidimensional nonlinear systems is also outlined in this section. Numerical results are provided in Section 4 and conclusions are drawn in Section 5.

2 Two-stage fourth-order time-accurate solver

In this section, we briefly review the two-stage fourth-order Lax-Wendroff type time discretization method [13]. Let us consider the following time-dependent nonlinear hyper-