# An All Speed Second Order IMEX Relaxation Scheme for the Euler Equations

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**Abstract.** We present an implicit-explicit finite volume scheme for the Euler equations. We start from the non-dimensionalised Euler equations where we split the pressure in a slow and a fast acoustic part. We use a Suliciu type relaxation model which we split in an explicit part, solved using a Godunov-type scheme based on an approximate Riemann solver, and an implicit part where we solve an elliptic equation for the fast pressure. The relaxation source terms are treated projecting the solution on the equilibrium manifold. The proposed scheme is positivity preserving with respect to the density and internal energy and asymptotic preserving towards the incompressible Euler equations. For this first order scheme we give a second order extension which maintains the positivity property. We perform numerical experiments in 1D and 2D to show the applicability of the proposed splitting and give convergence results for the second order extension.

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# 1   Introduction

We consider the non-dimensional Euler equations in $d$-space dimensions which are given by the following set of equations [1]

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$(\rho u)_t + \nabla \cdot \left( \rho \mathbf{u} \otimes \mathbf{u} + \frac{1}{M^2} p \mathbb{I} \right) = 0, \tag{1.1}$$
$$E_t + \nabla \cdot (\mathbf{u}(E+p)) = 0,$$

where the total energy is given by

$$E = \rho e + \frac{1}{2} M^2 \rho |u|^2 \tag{1.2}$$

and $e > 0$ denotes the internal energy. The density is denoted by $\rho > 0$, $\mathbf{u} \in \mathbb{R}^d$ is the velocity vector and $M$ is a given Mach number which controls the ratio between the velocity of the gas and the sound speed. Depending on the magnitude of the Mach number the characteristic nature of the flow changes. This makes the numerical simulation of these flows very challenging but also a very interesting research subject with a wide range of applications, for example in astrophysical stellar evolution or multiphase flows [2, 3]. For large Mach numbers, the flow is governed by compressible effects whereas in the low Mach limit the compressible equations converge towards the incompressible regime. This behaviour was studied for example in [1, 4, 5]. We refer to [6] for a study of the full Euler equations.

Standard schemes designed for compressible flows like the Roe scheme [7] or Godunov type schemes fail due to excessive diffusion when applied in the low Mach regime. A lot of work is dedicated to cure this defect, see for instance [1, 2, 8, 9]. Another way to ensure accurate solutions in the low Mach regime is the development of asymptotic preserving schemes which are consistent with its limit behaviour as $M$ tends to zero, see for example [10–12] and references therein.

Due to the hyperbolic nature of (1.1) the time step for an explicit scheme is restricted by a stability CFL condition that depends on the inverse of the fastest wave speed. In the case of (1.1) the acoustic wave speeds tend to infinity as $M$ tends to zero which leads to very small time steps to guarantee the stability of the explicit scheme. As a side effect all waves will be resolved by the explicit scheme although the fast waves are not necessary to capture the motion of the fluid as they carry a negligible amount of energy. Implicit methods on the other hand allow larger time steps but introduce diffusion on the slow wave which leads to a loss of accuracy. In addition at each iteration a non-linear often ill conditioned algebraic system has to be solved. Implicit-explicit (IMEX) methods try to overcome those disadvantages by treating the stiff parts implicitly and thus allow for a Mach number independent time step. Many of those schemes are based on a splitting of the pressure in the spirit of Klein [13] since the stiffness of the system is closely related with the pressure, see for example [10, 13, 14].