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Extended Synchronous Variational Integrators for Wave Propagations on Non-Uniform Meshes

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Abstract. This paper is concerned with the classical problem of wave propagation in discrete models of nonuniform resolution. We extend the traditional asynchronous variational integrators (AVIs) method to higher order and couple different spatial elements to adapt to nonuniform meshes. We show that the extension of AVIs method are stable, convergent and may reduce the spurious inter-grid reflection across meshes with different sizes. Numerical experiments are provided to verify the stability and convergence of the extended AVIs. The total energy is numerically conserved in our experiments.

AMS subject classifications: 35L05, 35A15, 65M22

Key words: Asynchronous variational integrators, nonuniform mesh, local time stepping, spurious reflections.

1 Introduction

The extension of the idea of variational integrators to the PDE context was made by Marsden et al. [1]. They showed, in a demonstration example, that the method was very promising for variational integrators in a spacetime context. As was mentioned, this approach was considerably developed, extended and applied in [2]. The asynchronous algorithms developed in that paper, and further in the present work, share many features in common with multi-time-step integration algorithms, sometimes termed subcycling methods. These algorithms have been developed in [3, 4], mainly to allow high frequency elements to advance at smaller time steps than the low-frequency ones. In its original version, the method grouped the nodes of the mesh and assigned to each group a

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different time step. Adjacent groups of nodes were constrained to have integer time-step ratios (see [4]), a condition that was relaxed in [3, 5]. Recently an implicit multi-time-step integration method was developed and analysed in [6]. We also mention the related work of Hughes and Liu [7] and Hughes [8] for the finite element frame. The freedom to choose the time step for each element, subject to stability considerations, as well as the way nodes are updated, are the distinguishing features of the asynchronous algorithms discussed in this paper.

The Asynchronous variational integrators [9], or AVIs for short, are distinguished by the following attributes: (i) The algorithms permit the selection of independent time steps in each element, and the local time steps need not bear an integral relation to each other; (ii) the algorithms derive from a spacetime form of a discrete version of Hamilton's variational principle. As a consequence of this variational structure, the algorithms conserve local momenta and a local discrete multisymplectic structure exactly.

The problem of wave propagation in discrete models of nonuniform spatial resolution has received considerable attention. Local time stepping for one-dimensional conservation laws was first proposed by Osher and Sanders [10]. Their schemes allow each element to take either an entire step or some fixed number of smaller steps. Another approach developed by Berger and Oliger [11] involves automatically taking smaller time steps where the mesh is refined. Most of the local time stepping techniques have been developed for fixed mesh structure. Tan et al. [12] combined the local time refinement techniques with the moving mesh methods. In the past two decades, there have been many efforts in developing efficient moving mesh algorithms, see, e.g. [13–16]. Qiao et al. [17, 18] concerned an adaptive time-stepping strategy which is unconditional energy stable. In [19], Kublik and Chopp developed the locally adaptive time stepping method for solving reaction-diffusion equations. In [20, 21], some recent developments, such as Runge-Kutta Nyström integrators and local discontinuous Galerkin method about nonlinear wave equation are proposed by Liu and Wu. It is also well known that wave reflection occurs at the interface of nonuniform meshes. Baz̆ant and his co-author in [22–24] found that using the small time step match with the minimum space step also causes some problems. When an elastic wave passes the different mesh sizes will cause the spurious reflection. More worse is that the spurious reflection is happening when the mesh size is much smaller than the wavelength. They only present the issue without solution. In [25], Venturini and Yang developed Replica time integrators (RTIs) allows for the two-way transmission of high-frequency harmonics across mesh interfaces. The two-way transmission afforded by RTIs is accomplished by representing the state of the coarse regions by means of replica ensembles, consisting of collections of identical copies of the coarse regions. The RTIs requires that the coarse meshes have to be an integer multiple of the fine meshes.

We extend the traditional AVIs method to high order elements which are adopted in the discretization of space for the simulation of linear wave propagation on non-uniform meshes. That means the coupling of elements with different order is considered. Since the elemental time step is determined by the size of the element, we use high order element