

## Super-Closeness between the Ritz Projection and the Finite Element Solution for some Elliptic Problems

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**Abstract.** We prove the super-closeness between the finite element solution and the Ritz projection for some second order and fourth order elliptic equations in both the  $H^1$  and the  $L^2$  norms. For the fourth order problem, a Ciarlet-Raviart type mixed formulation is used in the analysis. The main tool in the proof is a negative norm estimate of the Ritz projection, which requires  $H^{q+1}$  regularity for second order elliptic equations. Therefore the analysis is done on a domain  $\Omega$  with smooth boundary, and hence we only consider the pure Neumann boundary problems which can be discretized naturally on such domains, if ignoring the effect of numerical integrals. For the fourth order problem, our results amend the gap between the theoretical estimates and the numerical examples in a previous work [22].

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**Key words:** Finite element, Ritz projection, negative norm estimate, super-closeness estimate.

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## 1 Introduction

In a previous paper [22], the authors studied a Ciarlet-Raviart type mixed finite element method for a fourth order elliptic problem:

$$\begin{cases} \Delta^2 u - \nabla \cdot (a(x) \nabla u) + b(x)u = f, & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial \Delta u}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega. \end{cases}$$

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The above boundary condition was called “generalized Neumann boundary condition” in [12], and was discussed in [14, 22] for some fourth order elliptic and parabolic problems. It arises from the lubrication for thin viscous films and dislocation densities in plasticity (see [2, 13]). Optimal finite element error estimates were obtained for this problem, by using a novel super-closeness relation

$$\|P_h u - u_h\|_1 + \|P_h w - w_h\|_1 \leq Ch^{q+1} (\|u\|_{q+1} + \|w\|_{q+1}), \quad (1.1)$$

where  $w = -\Delta u$  is the dual variable,  $u_h, w_h$  are the finite element solution,  $P_h$  is a Ritz projection and  $q$  is the degree of polynomial defining the finite element spaces. However, the numerical results in [22] have shown a better performance than (1.1), indicating that (1.1) can possibly be improved.

Inspired by this problem, we study the super-closeness between the finite element solution and the Ritz projection for some second order and fourth order elliptic equations in both the  $H^1$  and the  $L^2$  norms. Such super-closeness results have helped to derive the error estimates in [22], and are also known to be essential in some finite element super-convergence analysis [18, 19]. They may also have important applications in the finite element approximations to some nonlinear partial differential equations, for example, the Cahn-Hilliard-type models [6, 8–10, 16, 21], the thin film models [5, 11, 15], and other related models [7, 17, 20]. We hope that our study will provide useful tools and insights into these problems.

The paper is organized as follows. In Section 2, we define the mesh, the finite element space, and the Ritz projection. Also presented is a negative norm estimate of the Ritz projection, which is essential to the proof of the super-closeness estimates. We then discuss the super-closeness for a second order elliptic problem, a simple fourth order problem and a more general fourth order problem, respectively, in Sections 3-5. Finally, numerical results are presented in Section 6.

## 2 Ritz projection and the negative norm estimate

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$  ( $d = 2$  or  $d = 3$ ) with smooth boundary. For  $K \subseteq \Omega$ , denote by  $W^{m,p}(K)$  the standard Sobolev space and by  $\|\cdot\|_{m,p,K}$ ,  $|\cdot|_{m,p,K}$  the norm and semi-norm on  $W^{m,p}(\Omega)$  (see [1]), respectively. When  $p = 2$ , the space  $W^{m,2}(K)$  is often written as  $H^m(K)$ , and its norm and semi-norm written simply as  $\|\cdot\|_{m,K}$  and  $|\cdot|_{m,K}$ . If  $K = \Omega$ , we further suppress the subscript  $K$  in the norms and semi-norms. Denote  $H^{-m}(\Omega) = (H^m(\Omega))'$ , which is different from the usual definition using the dual space of  $H_0^m(\Omega)$  but not uncommon in the literatures. Finally, we denote by  $\|\cdot\|$  and  $(\cdot, \cdot)$  the  $L^2$  norm and the  $L^2$  inner-product on  $\Omega$ , respectively.

Since the boundary of  $\Omega$  is smooth, we consider a partition  $\mathcal{T}_h$  of  $\Omega$  which allows curved boundary edges/faces. A 2D example is shown in Figure 1. To define the mesh, we first extend the meaning of “edge/face” to allow curved edges/faces, and consequently “simplex” to allow triangles/tetrahedra with one or more curved edges/faces.