Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0065

Vol. **28**, No. 3, pp. 1038-1084
September 2020

Analysis and Numerical Simulation of Hyperbolic Shallow Water Moment Equations

Julian Koellermeier[1,2,*] and Marvin Rominger[1]

[1] *Institut für Mathematik, Freie Universität Berlin, Berlin, Germany.*
[2] *School of Mathematical Sciences, Peking University, Beijing, China.*

Received 24 April 2019; Accepted (in revised version) 17 January 2020

Abstract. Shallow Water Moment Equations allow for vertical changes in the horizontal velocity, so that complex shallow flows can be described accurately. However, we show that these models lack global hyperbolicity and the loss of hyperbolicity already occurs for small deviations from equilibrium. This leads to instabilities in a numerical test case. We then derive new Hyperbolic Shallow Water Moment Equations based on a modification of the system matrix. The model can be written in analytical form and hyperbolicity can be proven for a large number of equations. A second variant of this model is obtained by generalizing the modification with the help of additional parameters. Numerical tests of a smooth periodic problem and a dam break problem using the new models yield accurate and fast solutions while guaranteeing hyperbolicity.

AMS subject classifications: 76D05, 35L65, 65M08

Key words: Shallow water equation, moment method, hyperbolicity.

1 Introduction

The shallow water equations (SWE) are widely used to describe flows where the vertical velocity components are much smaller than the horizontal components. Apart from hydrodynamic applications such as the computation of currents in Venice bay [32], the shallow water equations are used in many other scientific fields like weather forecasts [8], gravitational flow models like snow avalanches [7], and granular flows of liquids in the field of chemical engineering [9].

Nevertheless, the shallow water equations are often limited in their applications due to the loss of information by averaging the vertical velocity, see [18]. The assumption of a constant horizontal velocity with respect to height is a strong simplification and has been shown to be inaccurate in measurements [20,33].

*Corresponding author. *Email addresses:* `koellermeier@zedat.fu-berlin.de` (J. Koellermeier), `marvin.rominger@gmail.com` (M. Rominger)

A first approach to circumvent this problem is the multi-layer shallow water model, which yields a piecewise constant velocity profile to model changes in vertical direction, see e.g. [5, 14, 28, 35] for details on the model derivation, analysis, and numerical implementation. However, this introduces a number of difficulties, for example the coupling at the interfaces and hyperbolicity issues due to the discontinuous velocity description.

This paper investigates moment models for the shallow water equations, in which vertical changes of the velocity variable are allowed by adding further variables (so-called *moments*) such that deviations from that averaged constant velocity can be represented by a continuous polynomial. The moment method is already widely known in kinetic theory after the seminal work by H. Grad [16] and was later applied in channel flows [15, 37] and debris flows [26].

In [27] a moment method for the shallow water equations was introduced that succeeds in describing linear and quadratic velocity profiles for a smooth test case. We will use this model called Shallow Water Moment Equations (SWME) as a starting point of our investigations.

However, similar to the moment method in kinetic theory [2, 24, 39], hyperbolicity of the SWME is limited to a bounded domain in phase space. This lack of hyperbolicity was not discussed in detail in the shallow water moment model [27]. However, problems with hyperbolicity often lead to instabilities in other models, so that new hyperbolic models needed to be derived, for example [3, 31, 34, 38]. For rarefied gases the lack of hyperbolicity could be overcome using several strategies [4, 13, 22, 23], effectively resulting in the change of some terms in the higher order equations, but up to now no strategy to derive hyperbolic shallow water moment equations exists.

The aim of the paper is to investigate the hyperbolicity of the shallow water moment equations and systematically derive modified systems of equations that yield global hyperbolicity and sufficient accuracy in numerical simulations.

We first focus on the original SWME system and prove that loss of hyperbolicity occurs in practical applications, even when starting from initial conditions that are inside the hyperbolicity region. This might result in unpredictable behavior of the solution and makes it difficult to justify the use of the standard model. In one example we show emerging instabilities in a numerical solution.

The hyperbolic regularization is then based on a linearization of the SWME model around a linear deviation from equilibrium, similar to the models in [2] and [23]. The resulting system of equations is called Hyperbolic Shallow Water Moment Equations (HSWME). We investigate the structure and explicit terms in detail to allow for a global hyperbolicity proof for the high order system. Using the insights of this proof, it is possible to generalize the approach so that predefined propagation speeds can be obtained. This is achieved by the addition of some parameters β_i in the very last row of the system matrix, leading to a new hyperbolic system, called β-HSWME. The parameters are chosen so that the characteristic polynomial of the system matrix matches a given target polynomial. We show that there exists a unique solution and we give a practical example in which the solution is particularly simple to compute and only a single entry has to be