

## Mathematical Aspects Relative to the Fluid Statics of a Self-Gravitating Perfect-Gas Isothermal Sphere

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**Abstract.** In the present paper we analyze and discuss some mathematical aspects of the fluid-static configurations of a self-gravitating perfect gas enclosed in a spherical solid shell. The mathematical model we consider is based on the well-known Lane–Emden equation, albeit under boundary conditions that differ from those usually assumed in the astrophysical literature. The existence of multiple solutions requires particular attention in devising appropriate numerical schemes apt to deal with and catch the solution multiplicity as efficiently and accurately as possible. In sequence, we describe some analytical properties of the model, the two algorithms used to obtain numerical solutions, and the numerical results for two selected cases.

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**Key words:** Self-gravitating gas, Lane–Emden equation, multiple solutions.

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## 1 Introduction

The motivation behind our research in gravitational fluid dynamics is described in details in the introduction to our recent paper [12]. The core purpose of that paper was to study the static configurations of a self-gravitating isothermal sphere (see Fig. 1). The emphasis therein was mainly on physics, fluid statics and thermodynamics in particular; we thoroughly described the adopted physical model and the ensuing governing equations,

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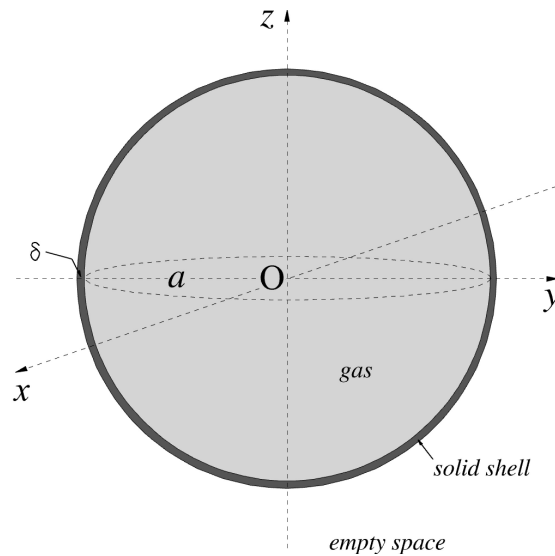


Figure 1: Study case: fluid statics of a self-gravitating isothermal perfect gas inside a spherical solid shell with internal radius  $a$  and thickness  $\delta$ .

and discussed numerical results. The spherically symmetric mathematical problem inside the gas turns out to hinge completely on the solution of the isothermal Lane-Emden equation [10]

$$\frac{d^2 \log \rho}{dr^2} + \frac{2}{r} \frac{d \log \rho}{dr} + \frac{4\pi G}{RT} \rho = 0, \quad (1.1)$$

which provides the mass-density radial profile. Once this is available, pressure and gravitational field follow straightforwardly from

$$p = \rho RT \quad (1.2)$$

and

$$g = RT \frac{d \log \rho}{dr}. \quad (1.3)$$

The following notation is used in Eqs. (1.1)-(1.3):

- $r$  radial distance from origin
- $\rho$  mass density
- $G$  gravitational constant,  $6.67428 \cdot 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
- $T$  temperature
- $R$  gas constant
- $p$  pressure
- $g$  gravitational field

Eq. (1.1) is complemented with two conditions. The condition

$$\left. \frac{d \log \rho}{dr} \right|_{r=0} = 0 \quad (1.4)$$