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A Lowest-Order Mixed Finite Element Method for the Elastic Transmission Eigenvalue Problem

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Abstract. The goal of this paper is to develop numerical methods computing a few smallest elastic interior transmission eigenvalues, which are of practical importance in inverse elastic scattering theory. The problem is challenging since it is nonlinear, non-self-adjoint, and of fourth order. In this paper, we construct a lowest-order mixed finite element method which is close to the Ciarlet-Raviart mixed finite element method. The scheme is based on Lagrange finite element and is one of the less expensive methods in terms of the amount of degrees of freedom. Due to the non-self-adjointness, the discretization of elastic transmission eigenvalue problem leads to a non-classical mixed problem which does not fit into the framework of classical theoretical analysis. Instead, we obtain the convergence analysis based on the spectral approximation theory of compact operator. Numerical examples are presented to verify the theory. Both real and complex eigenvalues can be obtained.

AMS subject classifications: 65N25, 65N30, 47B07

Key words: Elastic transmission eigenvalue problem, mixed finite element method, Lagrange finite element.

1 Introduction

Transmission eigenvalue problem is very important in the qualitative reconstruction in the inverse scattering theory of inhomogeneous media. For example, the eigenvalues can be used to estimate the physical properties of scattering object [6, 30]. The transmission

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eigenvalue problem is non-self-adjoint and is not covered by the standard theory of partial differential equation. It is numerically challenging because of the nonlinearity and the complicated spectrum without a priori information. In most cases, the continuous problem is degenerate with an infinite dimensional eigenspace associated with the zero eigenvalue, which has no physical meaning and makes it difficult to be solved. There are different types of transmission eigenvalue problems, such as acoustic transmission eigenvalue problem, electromagnetic transmission eigenvalue problem, elastic transmission eigenvalue problem, etc.

Since 2010, effective numerical methods for the acoustic transmission eigenvalues have been developed by many researchers [1, 7, 8, 13, 14, 17, 18, 20–22, 24, 29, 32, 35, 36]. There are much fewer works for the electromagnetic transmission eigenvalue problem [16, 27, 31]. In this paper, we try to develop effective numerical methods for transmission eigenvalue problem of elastic waves. Compared with the acoustic transmission eigenvalue problem, the eigenfunctions are vectors which make it more difficult to design convergent methods. There exist very limited numerical methods for elastic transmission eigenvalue problem. To the best of our knowledge, there are only two works on numerical algorithms. In [19], the elastic transmission eigenvalue problem is reformulated as the combination of a nonlinear function and a series of fourth order self-adjoint eigenvalue problems. The value of the nonlinear function corresponds to the generalized eigenvalue of a fourth order self-adjoint eigenvalue problem which can be discretized by H^2 conforming finite element methods. The roots of the nonlinear function are the transmission eigenvalues. The authors apply the secant iterative method to compute the transmission eigenvalues. However, at each step, a fourth order self-adjoint eigenvalue problem needs to be solved and only real eigenvalues can be captured. In [34], an interior penalty discontinuous Galerkin method using C^0 Lagrange elements (C^0IP) is proposed for the elastic transmission eigenvalue problem. They are simpler than C^1 elements and come in a natural hierarchy. It's much easier to be implemented. However, when the polynomial degree p increases, the degrees of freedom increase remarkably. Although the existence of elastic transmission eigenvalues is beyond our concern, we want to remark that there exist only a few studies on the existence of the elasticity transmission eigenvalues [3, 4, 9, 10]. We hope that the numerical results can give some hints on the analysis of the elasticity transmission eigenvalue problem.

In this paper, we construct a mixed finite element method for the elastic transmission eigenvalue problem. For acoustic transmission eigenvalue problem, the related works for mixed element method can be referred to [8, 13, 20, 35, 36]. The mixed scheme in [8, 20] which is close to the Ciarlet-Raviart discretization of biharmonic problem is based on Lagrange finite element method. For the nonzero transmission eigenvalues, this scheme is equivalent to the one proposed in [13]. However, the scheme in [8, 20] can eliminate the zero transmission eigenvalue which has an infinite dimensional space and has no physical meaning. A mixed formulation in terms of three scaler fields and a spectral-mixed method are constructed in [36]. In [35], the authors propose a multi-level mixed formulation in terms of seven scalar fields. An equivalent linear mixed formulation of