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Weak Galerkin and Continuous Galerkin Coupled Finite Element Methods for the Stokes-Darcy Interface Problem

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Abstract. We consider a model of coupled free and porous media flow governed by Stokes equation and Darcy's law with the Beavers-Joseph-Saffman interface condition. In this paper, we propose a new numerical approach for the Stokes-Darcy system. The approach employs the classical finite element method for the Darcy region and the weak Galerkin finite element method for the Stokes region. We construct corresponding discrete scheme and prove its well-posedness. The estimates for the corresponding numerical approximation are derived. Finally, we present some numerical experiments to validate the efficiency of the approach for solving this problem.

AMS subject classifications: 65N30, 65N15, 65N12, 74N20

Key words: Finite element methods, weak Galerkin finite element methods, weak gradient, Stokes equations, Darcy's equation.

1 Introduction

The coupled model of fluid flow with porous media flow has gained increasing attention in recent years with many applications, such as groundwater flows [10,14,26], industrial filtrations [17], flow in vuggy porous media [2,3] etc. In the model, the Stokes equation is used to describe the free flow and the Darcy's law is used to describe the flow in porous media. Due to the different mechanism of flow in subdomains, it is necessary to introduce appropriate interface conditions on the interface between the two different flows. Firstly, the normal velocity is continuous, which is the result of mass conservation. Secondly,

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the normal force on the interface is balanced. Thirdly, the slip speed is proportional to the tangential tensor. The third condition is generally referred to as the Beavers-Joseph interface condition, which was first observed through experiments [7]. In this paper, we utilize the simplified case: the Beavers-Joseph-Saffman (BJS) interface condition [21, 22, 37]. In this Beavers-Joseph-Saffman condition, the contribution of the tangential velocity of the porous media flow is neglected.

Various numerical methods have been proposed for the Stokes-Darcy system. In [28], continuous finite element method and mixed finite method are considered in the Stokes and Darcy subdomains, respectively. The authors also proved existence and uniqueness of weak solutions to the problem. Continuous finite elements used in both regions are studied in [14]. In [35], discontinuous Galerkin (DG) method is employed in the Stokes region while the mixed finite element method is employed in Darcy region. The analysis of DG methods in two subdomains can be found in [36]. In [16, 20], HDG methods are considered to deal with the problem. Domain decomposition approaches make it possible to break up the coupled problem into two single problems, which might be solved in parallel. The related work can be found in [11, 15]. The results in [4, 31] demonstrate that the multi-grid approach can be applied to this problem. In recent years, people pay attention to numerical schemes on general meshes because of its flexibility in practice. Corresponding works for Stokes-Darcy system include virtual element method [43], weak Galerkin finite element methods [12, 23, 24] and so on.

The weak Galerkin (WG) finite element method is first proposed by Wang and Ye for solving the second order elliptic equation in [39]. The key idea of this method is that the classical derivatives were substituted by weakly defined derivatives for totally discontinuous functions. The continuity of the approximating function is ensured by adding a stabilizer. In form, WG method is closely related to the framework of HDG method. The relationship of WG method and HDG method has been discussed in [9, 42] so we will not extend here. In recent years the WG method is widely used because of its flexibility, such as Stokes equation [34, 41, 46, 48], Brinkman equation [30, 47, 49], Darcy equation [25, 27, 29], convection problems [18, 19], elasticity problems [44, 45], elliptic interface problems [13, 32, 33]. Recently, WG methods for the Stokes-Darcy system is researched. In [12], the authors consider the mixed formulation in the Darcy region, both the Stokes equation and Darcy equation involves the flux and the pressure. In addition, the formulation of the Stokes equations involves symmetric full stress tensor. In [23], authors used the Stokes equations in velocity -pressure formulation and the Darcy equation in primal pressure formulation, the approximations of the velocity, hydraulic and pressure are piecewise constant. The same formulation of the Stokes-Darcy system was discussed in [24], the authors use different degrees of polynomials on the arbitrary polygonal meshes and gave a series of numerical experiments.

Continuous finite elements have the benefit of simplicity and there are many codes available [38]. Employing the continuous finite element for the Darcy equation in the second order primal formulation makes the porous medium simulators highly optimized [35] and does not create additional degrees of freedom. The WG numerical scheme for