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Recurrence Phenomenon for Vlasov-Poisson Simulations on Regular Finite Element Mesh

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Abstract. In this paper, we focus on one difficulty arising in the numerical simulation of the Vlasov-Poisson system: when using a regular grid-based solver with periodic boundary conditions, perturbations present at the initial time artificially reappear at a later time. For regular finite-element mesh in velocity, we show that this recurrence time is actually linked to the spectral accuracy of the velocity quadrature when computing the charge density. In particular, choosing trigonometric quadrature weights optimally defers the occurrence of the recurrence phenomenon. Numerical results using the Semi-Lagrangian Discontinuous Galerkin and the Finite Element/Semi-Lagrangian method confirm the analysis.

AMS subject classifications: 65M22, 65T40, 82D10

Key words: Finite element mesh, Vlasov-Poisson system, Semi-Lagrangian Discontinuous Galerkin method, trigonometric quadrature.

1 Introduction

Accurate long run simulations of plasmas (gas of charged particles) are crucial in many applications like energy production. In this work, we are mainly interested in collisionless plasmas, whose dynamics can be described by a Vlasov-Poisson system under some suitable physical assumptions. The Vlasov equation is a transport equation satisfied by the kinetic distribution function of the particles in phase-space (position/velocity). Developing efficient and accurate numerical schemes remains a real challenge. Among several difficulties, one of them appears when solving the kinetic transport equations on a

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regular phase-space grid with periodic boundary conditions: a recurrence phenomenon occurs and suddenly destroys the accuracy of the numerical solution. In this paper, we show that the recurrence time is linked to the precision of the velocity quadrature when computing the charge density. In particular, choosing trigonometric quadrature weights optimally weaken the recurrence phenomenon.

The recurrence phenomenon is for the first time mentioned in the seminal paper of Cheng and Knorr [6], where they consider a regular grid of the phase space and a semi-Lagrangian scheme to solve the Vlasov equation. When considering periodic boundary conditions, the Vlasov dynamics exhibit phase mixing: small initial spatial perturbations result in oscillations in the velocity distribution, whose frequency grows in time. Consequently, at some time, the velocity mesh cannot capture these oscillations anymore. This leads to an aliasing effect: when charge density is computed, the large frequency oscillations are numerically interpreted as small frequency ones. Note that charge density and the electric field are related through the Poisson equation and thus the recurrence phenomenon is rather observed on the electric energy time evolution. When using a uniform grid, the recurrence time can be explicitly given in this simple case: $T=2\pi/(k\Delta v)$, where k is the spatial perturbation mode and Δv is the velocity mesh size.

Vlasov solvers based on a phase space grid enable one to reach high order accuracy. This is in contrast with particle methods [20], which do not involve a velocity grid but are subject to statistical noise. The semi-Lagrangian method, used in [6], is a grid-based transport solver: each time step consists in computing the characteristics of the transport equation backward and interpolating the nodal values of the distribution function at the feet of the characteristics. Numerous variants of the methods have been studied, considering different interpolation operators (e.g. spline, Lagrange, Hermite interpolation), conservative version of the method. We refer to [27] and references therein. Their main advantage is to overcome the CFL stability condition. Another class of grid-based methods are the Eulerian one, like finite-volume [16] or discontinuous Galerkin method [4,18]. These methods handle more naturally complex geometries at the price of satisfying the CFL condition.

High-order accurate grid-based solvers are designed to carry out long-run simulations and recurrence phenomenon is a real limitation. The simplest way to postpone the recurrence time is to refine the mesh by taking a smaller Δv. However, it often demands too much computation time and storage resources. In consequence, several works have considered methods to suppress or at least reduce the recurrence phenomenon. The main method consists in using filtering techniques. This can be done by adding an artificial diffusion to the Vlasov equation [21], adding outflow boundary conditions in the Fourier velocity variable [13–15] or using cutoff procedures in Fourier space [12]. For the latter method, the influence of the cutoff on the invariants of the dynamics, such as mass, entropy or L^p norms, has been studied in [10]. Filtering techniques have also been studied and analysed in the context of the Hermite-spectral method [5]. In another direction, randomization of the velocity points in each velocity cells have been considered in [1].