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A Well-Balanced Gas Kinetic Scheme for Navier-Stokes Equations with Gravitational Potential

Songze Chen[1,*], Zhaoli Guo[1] and Kun Xu[2]

[1] *State Key Laboratory of Coal Combustion, School of Energy and Power Engineering, Huazhong University of Science and Technology, Wuhan, China.*
[2] *The Hong Kong University of Science and technology, Clear Water Bay, Kowloon, Hong Kong, China.*

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Abstract. The hydrostatic equilibrium state is the consequence of the exact balance between hydrostatic pressure and external force. Standard finite volume cannot keep this balance exactly due to their unbalanced truncation errors. In this study, we introduce an auxiliary variable which becomes constant at isothermal hydrostatic equilibria and propose a well-balanced gas kinetic scheme for the Navier-Stokes equations. Through reformulating the convection term and the force term via the auxiliary variable, zero numerical flux and zero numerical source term are enforced at the hydrostatic equilibrium state instead of the balance between hydrostatic pressure and external force. Several problems are tested to demonstrate the accuracy and the stability of the new scheme. The results confirm that, the new scheme can preserve the exact hydrostatic solution. The small perturbation riding on hydrostatic equilibria can be calculated accurately. More importantly, the new scheme is capable of simulating the process of converging towards hydrostatic equilibria from a highly unbalanced initial condition. The ultimate state of zero velocity and constant temperature is achieved up to machine accuracy. As demonstrated by the numerical experiments, the current scheme is very suitable for small amplitude perturbation and long time running under gravitational potential.

AMS subject classifications: 76M99, 86A10

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1 Introduction

Gravity is involved in many physical problems, including astrophysical problems like core-collapse supernova, atmospheric motions on planet, smoke stratification in com-

*Corresponding author. *Email addresses:* jacksongze@hust.edu.cn (S. Chen), zlguo@hust.edu.cn (Z. Guo), makxu@ust.hk (K. Xu)

partment fires etc. In order to understand these phenomena and make reliable prediction, conservation laws with gravitational force are invoked in the form of partial differential equations. However, numerical simulations of these systems are not easy from the following aspects: (1) for a long time evolution, the truncation error will accumulate and dramatically affect the final solution of an isolated gravitational system [1,2]; (2) to predict small perturbation, say, numerical weather prediction and climate modeling, the truncation error will mask the small perturbations on the top of stationary solution [2,3].

These two problems can be attributed to unbalanced discretization of the convection term and gravitational force term. Consider a fluid system under gravity governed by the Euler equations. The fluid system possesses a stationary state known as the hydrostatic equilibrium state in which the gravitational force is exactly balanced by the pressure gradient. However, in conventional numerical schemes, the gravitational force term and the convection term are discretized separately, thereby, the truncation errors cannot cancel each other. As a result, the conventional numerical schemes are not able to preserve the hydrostatic equilibrium state, and may induce unacceptable spurious motions [4]. In this context, a numerical scheme which ensures the hydrostatic balance exactly on discrete level is termed a well-balanced scheme.

Many well-balanced schemes has been developed, especially for shallow water equations [5–7]. But the techniques developed for shallow water equations seem not easy to be implemented in the Euler equations with gravitational force term. This problem bothers the CFD community for a long time.

Botta et al. [4] developed a well-balanced finite volume method for the Euler equations, using a discrete Archimedes principle to express the gravity source term as the cell surface integral of the reconstructed hydrostatic pressure.With the help of kinetic theory, Xu et al. [8] proposed a kind of well-balanced scheme for the Euler equations in 2010. The gravitational potential is approximated as a step function inside each cell, and the amount of particle penetration and reflection from the cell interface is evaluated according to the incident particle velocity and the strength of the potential barrier at the cell interface. This scheme can maintain hydrostatic equilibrium state exactly if the numerical integration used in kinetic flux is evaluated accurately.

In recent years, the development of well-balanced schemes for the Euler equations has gained more attention. In 2012, Xing and Shu [2] developed a special source term discretization so that the resulting WENO scheme balances the zero-velocity and constant temperature steady state solutions to machine accuracy, and at the same time maintains the high order accuracy and essentially non-oscillatory property for general solutions. Unlike the kinetic scheme with step potential assumption, their scheme is free of analytical or numerical integration. Käppeli and Mishra [9] developed a novel reconstruction of the enthalpy which is based on local constant entropy assumption. After that they proposed a more general pressure reconstruction using a local analytical integration of hydrostatic equation, and demonstrated the efficiency of their well-balanced schemes for a broad set of astrophysical scenarios with several types of equation of state [10]. Ghosh