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Central Discontinuous Galerkin Methods for the Generalized Korteweg-de Vries Equation

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Abstract. In this paper, we develop central discontinuous Galerkin (CDG) finite element methods for solving the generalized Korteweg-de Vries (KdV) equations in one dimension. Unlike traditional discontinuous Galerkin (DG) method, the CDG methods evolve two approximate solutions defined on overlapping cells and thus do not need numerical fluxes on the cell interfaces. Several CDG schemes are constructed, including the dissipative and non-dissipative versions. L^2 error estimates are established for the linear and nonlinear equation using several projections for different parameter choices. Although we can not provide optimal *a priori* error estimate, numerical examples show that our scheme attains the optimal $(k+1)$-th order of accuracy when using piecewise k-th degree polynomials for many cases.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Kortewegde Vries equation, central DG method, stability, error estimates.

1 Introduction

In this paper, we develop central discontinuous Galerkin (CDG) finite element methods for solving the generalized Korteweg-de Vries (KdV) [23] equation

$$u_t+f(u)_x+\sigma u_{xxx}=0, \tag{1.1}$$

where σ is a given constant and f is a smooth function. We will focus on one-dimensional case in this paper, however, the numerical methodologies can be generalized to multi-dimensional case.

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The KdV type equations and similar models that feature nonlinearity and dispersion arise as mathematical models for the propagation of physical waves in a wide variety of situations, e.g. [1, 2, 6, 7, 10, 20, 39]. They play an important role in applications, such as fluid mechanics [11, 37], nonlinear optics [3, 18], acoustics [38, 44], and plasma physics [41, 43]. They also have an enormous impact on the development of nonlinear mathematical science and theoretical physics. Due to their importance in applications and theoretical studies, various numerical methods have been proposed and used in practice to solve this type of equation, e.g. [8, 9, 21]. In many situations, such as in the quantum hydrodynamic models of semiconductor device simulations [17] and in the dispersive limit of conservation laws [25], the third order derivative terms might have small or even zero coefficients in some parts of the domain, which means the equation is "convection dominated". The design of stable, efficient and high order methods, especially those for the convection dominated cases is a big challenge. Hence the discontinuous Galerkin (DG) method is a good choice because of its advantages in dealing with convection terms. The DG methods for KdV type equations have developed significantly in recent years. The first DG method, which is the local discontinuous Galerkin (LDG) method, for the KdV equation was introduced by Yan and Shu in [49]. The main idea of the LDG method is to introduce new auxiliary variables and rewrite the original equation into several first order equations. Later the LDG method was analyzed in [19, 29, 45, 46] and recently energy conserving LDG methods for wave equations are popular [12, 50]. In [14], a DG method, the so-called ultra-weak DG method [40] for the KdV equation was devised by using the ultra-weak formulation, which avoids the introduction of auxiliary variables and equations. Recently, several conservative DG methods [5, 13, 16, 22] were developed for KdV type equations to preserve quantities such as the mass and the L^2-norm of the solutions.

On the other hand, the central scheme of Nessyahu and Tadmor [36] computes hyperbolic conservation laws on a staggered mesh and avoids the Riemann solver. In [24], Kurganov and Tadmor introduce a new kind of central scheme without the large dissipation error related to the small time step size by using a variable control volume whose size depends on time step size. Liu [31] uses another coupling technique to avoid the excessive numerical dissipation for small time steps. The overlapping cells evolves two independent overlapping subcell averages for each given cell, which opens up many new possibilities. The advantages of overlapping cells motivates the combination of the central scheme and the DG method, which results in the CDG methods [30, 32, 33]. The CDG method evolves two copies of approximating solutions defined on staggered meshes and avoids using numerical fluxes which can be complicated and costly [26]. Like some previous central schemes, the CDG method also avoids the excessive numerical dissipation for small time steps by a suitable choice of the numerical dissipation term. Besides, the central method carry many features of standard DG methods, such as compact stencil, easy parallel implementation, etc. It is generally understood that the CDG method allows for a larger CFL number compared to the regular DG method [33]. Later in [34], the central local discontinuous Galerkin (CLDG) method was introduced to solve dif-