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The Collocation Basis of Compact Finite Differences for Moment-Preserving Interpolations: Review, Extension and Applications

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Abstract. The diagnostic of the performance of numerical methods for physical models, like those in computational fluid mechanics and other fields of continuum mechanics, rely on the preservation of statistical moments of extensive quantities. Dynamic and adaptive meshing often use interpolations to represent fields over a new set of elements and require to be conservative and moment-preserving. Denoising algorithms should not affect moment distributions of data. And numerical deltas are described using the number of moments preserved. Therefore, all these methodologies benefit from the use of moment-preserving interpolations. In this article, we review the presentation of the piecewise polynomial basis functions that provide moment-preserving interpolations, better described as the collocation basis of compact finite differences, or Z-splines. We present different applications of these basis functions that show the improvement of numerical algorithms for fluid mechanics, discrete delta functions and denoising. We also provide theorems of the extension of the properties of the basis, previously known as the Strang and Fix theory, to the case of arbitrary knot partitions.

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1 Introduction

Conservative interpolations are often necessary for the improvement of numerical algorithms in which the discretisation mesh is dynamic and changes, moves or adapts in

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time [1–5]. The interpolation provides the values of the discretised fields on new elements and therefore it must at least preserve the volume integral of extensive quantities like mass, momentum and energy, among others. Formulas for conservative interpolations have been provided by several authors in control volume schemes [6–8], arbitrary Lagrangian-Eulerian [9–18], semi-Lagrangian [19] and moving mesh methods [20, 21].

The concept of conservative interpolations can be further extended to the preservation of more statistical moments, which have often some very important physical meaning. For example, for the approximation of analytic functions, such that as many of the moments of the approximation are the same as those of the function [22]. Moment-preserving interpolations, applied to a sampled or discretised extensive field, are desirable for the implementation of numerical methods in which interpolations could modify higher-order moments, thus introducing undesirable effects related to them, like spurious forces and torques, or artificial changes in total energy. In fact, many diagnostics for numerical methods are based on the computation of statistical moments and therefore, moment-preserving interpolations are crucial for the design of high quality schemes [2].

For a complete chronological overview of the developments in interpolation theory until 2002, see [23]. Crucial to our exposition are the works of Whittaker [24] and Shannon [25], in which the conditions for exact approximations were given and the *sinc* function was identified as the perfect cardinal interpolation kernel. The works of Hermite [26] and Birkhoff [27], who studied the polynomials that satisfy special criteria concerning the value of a function and/or the value of any of its derivatives in any given points. And the work of Bessel [28], who proposed a close relative to the moment-preserving basis, the well known Bessel interpolation, constructed using derivative values approximated with central differences and Hermite interpolation.

Other concepts significant to our exposition are those studied intensively during the past decades, primarily in approximation theory of convolution-based interpolations [23]. More specifically, the importance of the moment-preserving property was clarified in the theory developed by Strang and Fix [23, 29], in which the following conditions were proven to be equivalent: 1) The discrete convolution kernel ϕ has Lth-order zeroes in the Fourier domain, 2) The kernel ϕ is capable of reproducing all monomials of degree $n \leq L-1$, 3) The first L discrete moments of the kernel ϕ are constants, and 4) For each sufficiently smooth function f, there is a real constant C such that the L_2 norm of the difference between f and its approximation f_T is bounded by $||f-f_T||_{L_2} \leq CT^L||f^(L)||_{L_2}$ as $T \to 0$.

The moment-preserving basis was finally presented in 2003 as an internal ETH Zurich report [30] under the name of Z-splines. The basis is closely related to the Bessel splines, but they are the piecewise Hermite interpolation polynomials with compact finite differences approximating derivatives, instead of central difference approximations. Therefore, the Z-splines can be better described as the collocation basis of compact finite differences. Early attempts to publish the basis found fierce resistance from referees that didn't have enough elements to know if the basis was new or something forgotten and already published. The exposition of the basis was further improved in a PhD thesis in