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H^2-Conforming Methods and Two-Grid Discretizations for the Elastic Transmission Eigenvalue Problem

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Abstract. The elastic transmission eigenvalue problem has important applications in the inverse elastic scattering theory. Recently, the numerical computation for this problem has attracted the attention of the researchers. In this paper, we propose the H^2-conforming methods including the classical H^2-conforming finite element method and the spectral element method, and establish the two-grid discretization scheme. Theoretical analysis and numerical experiments show that the methods presented in this paper can efficiently compute real and complex elastic transmission eigenvalues.

AMS subject classifications: 65N25, 65N30

Key words: Elastic transmission eigenvalues, linear weak formulation, finite element, spectral element, the two-grid discretization, error estimates.

1 Introduction

Transmission eigenvalues including acoustic, electromagnetic and elastic transmission eigenvalue not only have wide physical applications, for example, they can be used to obtain estimates for the material properties of the scattering object [1–4], but also have theoretical importance in the uniqueness and reconstruction in the inverse scattering theory [5].

In the last decade, numerical methods for the acoustic and the electromagnetic transmission eigenvalues have been developed by many researchers, see books [6,7] and references cited therein, and the articles [8–13]. Recently, the computation of the elastic transmission eigenvalues has also attracted the attention of the researchers: Xi et al. [14] studied an interior penalty discontinuous Galerkin method using the C^0 Lagrange elements

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(C^0IP method); Xi and Ji [15] studied a lowest-order mixed finite element method which is close to the Ciarlet-Raviart mixed finite element method; Ji et al. [16] apply the secant iterative method to compute the real eigenvalues in which a fourth order self-adjoint eigenvalue problem needs to be solved by the H^2-conforming finite element method at each iterative step. Inspired by these works, in this paper, we further study the H^2-conforming methods including the classical H^2-conforming finite element method and the H^2-conforming spectral element method, and study the two-grid method for the elastic transmission eigenvalue problem. Our work has three features as follows:

(1) This work is challenging since the fact that the underlying interior elastic transmission eigenvalue problem is quadratic in the eigenvalue parameter, nonselfadjoint, of fourth order, and the eigenfunctions are vectors. We use the linearization technique in [17] to transform the elastic formulation (2.1)-(2.4) as a nonselfadjoint linear eigenvalue problem (2.10) whose left-hand side term is a selfadjoint, continuous and coercive sesquilinear form. Based on the resulting linear weak formulation, we propose the H^2-conforming methods including the classical H^2-conforming finite element method and the spectral element method. Our discretization scheme (see (3.2) or (5.1)) has a positive definite Hermitian and block diagonal stiff matrix, thus the computation of the eigenpairs for (5.1) is efficient. We use Babuska-Osborn's spectral approximation theory [18] to give the error analysis for the H^2-conforming method. For the acoustic transmission eigenvalue problem, the related works for the H^2-conforming element method can be referred to [19–22]. Compared with the acoustic transmission eigenvalue problem, the acoustic eigenfunctions are scalars while the elastic eigenfunctions are vectors, so the error analysis and numerical realization of the methods for the elastic problem is somewhat difficult and more efforts are needed.

(2) The efficient spectral-Galerkin method for fourth order equations has been developed (see, e.g., [23–26]), and it has been applied to the acoustic transmission eigenvalue problem (see, e.g., [27,28]). Based on the linear weak formulation (2.10), in this paper, we propose an H^2-conforming spectral-element method for the elastic transmission eigenvalue problem, and obtain some numerical eigenvalues of superior accuracy.

(3) In 1992, for the nonsymmetric or indefinite problems Xu [29] introduced a two-grid discretization technique which has been successfully applied to eigenvalue problems later (see, e.g., [30–38] and the references therein). In this paper, based on the weak formulation (2.10) we study a two-grid discretization scheme to solve the elastic transmission eigenvalue problem. With the scheme, the solution of the eigenvalue problem on a fine grid π_h is reduced to the solution of the primal and dual eigenvalue problem on a much coarser grid π_H and the solutions of two linear algebraic systems with the same positive definite Hermitian stiff matrix, and the degrees of freedom of the linear algebraic systems is only half of the degrees of freedom of the eigenvalue problem on the fine grid π_h (see Remark 4.2), and the resulting solution still maintains an asymptotically optimal accuracy.

In this paper, C denotes a positive constant independent of the mesh size h, and we use the symbol $a \lesssim b$ to mean that $a \leq Cb$.