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An Efficient Finite Element Method with Exponential Mesh Refinement for the Solution of the Allen-Cahn Equation in Non-Convex Polygons

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Abstract. In this paper we consider the numerical solution of the Allen-Cahn type diffuse interface model in a polygonal domain. The intersection of the interface with the re-entrant corners of the polygon causes strong corner singularities in the solution. To overcome the effect of these singularities on the accuracy of the approximate solution, for the spatial discretization we develop an efficient finite element method with exponential mesh refinement in the vicinity of the singular corners, that is based on $(k-1)$-th order Lagrange elements, $k\geq 2$ an integer. The problem is fully discretized by employing a first-order, semi-implicit time stepping scheme with the Invariant Energy Quadratization approach in time, which is an unconditionally energy stable method. It is shown that for the error between the exact and the approximate solution, an accuracy of $\mathcal{O}(h^k+\tau)$ is attained in the L^2-norm for the number of $\mathcal{O}(h^{-2}\ln h^{-1})$ spatial elements, where h and τ are the mesh and time steps, respectively. The numerical results obtained support the analysis made.

AMS subject classifications: 65M50, 65M60, 65M15, 65Z05

Key words: Allen-Cahn equation, non-convex polygon, mesh refinement, corner singularities, finite element method, invariant energy quadratization, error estimation.

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1 Introduction

It is well-known that the solutions of nonlinear diffusion equations which have very small diffusion coefficients or very large reaction terms often develop internal transition layers, called interfaces, that separate the spatial domain into different phase regions.

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Moreover, when these problems are posed in two-dimensional, non-convex polygonal domains, the solutions can also exhibit corner singularities. The classical finite-difference and finite element methods become ineffective around the singular corners and methods with special constructions are required for highly-accurate solutions, for which knowledge of the nature of the corner singularities becomes crucial.

An example of this is the Allen-Cahn equation introduced in [1], namely

$$u_t - \Delta u = \frac{1}{\epsilon^2} f(u), \tag{1.1}$$

which is a simple model of evolution of antiphase boundaries, where $\epsilon > 0$ is a small parameter and $f(u)$ is a bistable nonlinearity. The Allen-Cahn equation has been widely used to model various phenomena in nature. In particular, it has become a basic model equation for the diffuse-interface approach developed to study phase transitions and interfacial dynamics in material science [2]. Starting from arbitrary initial data, the solution of equation (1.1) develops interior layers, or interfaces. On one side of the interface, $u \sim u^+$ and on the other side $u \sim u^-$. The stable solution corresponds to an interface with a minimal perimeter that intersects the sides of the boundary orthogonally [3,4].

Due to the nonlinearity in the equation, the solution of the Allen-Cahn equation can only be sought numerically. However, numerical methods regarding the solution of this equation have been largely considered in convex polygonal domains. To name a few of these studies, in [5] details were provided about the effectiveness of the high order and adaptive discretization schemes and the desirable choices of discretization parameters for simulations with very small interfacial width ϵ, a brief review and a critical comparison of the performance of several numerical schemes for solving the Allen-Cahn equation is presented in [6], and in [7] the error estimates for selected schemes with a spectral-Galerkin approximation for the numerical solution of the Allen-Cahn equation is analysed. However most of these methods cannot be directly applied in domains with re-entrant corners due to the possible low regularity of the solution at the intersection of the interface with the corners.

In this paper, we consider the numerical solution of the Allen-Cahn equation (1.1) in non-convex polygons. To overcome the effect of the corner singularities on the accuracy of the approximate solution, exponentially refined polar meshes are constructed in the vicinity of the corners of the polygon for the spatial finite element mesh. The proposed local mesh refinement is exponential in the polar radius r, uniform in the polar angle θ, and connected with the mesh in the remainder of the domain so that no additional techniques are required for coupling the solution in the subdomains. We obtain the numerical solution on the constructed mesh by using the finite element method based on $(k-1)$-th order Lagrange elements in space, $k \geq 2$, for all $t \geq 0$.

To fully discretize the problem, we consider the use of an unconditionally energy stable scheme for the time-stepping discretization. When the underlining energy law is stable in the fully discretized equation, it is generally possible to use a relatively coarse mesh in the simulation. Consequently such a method can reduce the cost of computa-