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Effective Two-Level Domain Decomposition Preconditioners for Elastic Crack Problems Modeled by Extended Finite Element Method

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Abstract. In this paper, we propose some effective one- and two-level domain decomposition preconditioners for elastic crack problems modeled by extended finite element method. To construct the preconditioners, the physical domain is decomposed into the "crack tip" subdomain, which contains all the degrees of freedom (dofs) of the branch enrichment functions, and the "regular" subdomains, which contain the standard dofs and the dofs of the Heaviside enrichment function. In the one-level additive Schwarz and restricted additive Schwarz preconditioners, the "crack tip" subproblem is solved directly and the "regular" subproblems are solved by some inexact solvers, such as ILU. In the two-level domain decomposition preconditioners, traditional interpolations between the coarse and the fine meshes destroy the good convergence property. Therefore, we propose an unconventional approach in which the coarse mesh is exactly the same as the fine mesh along the crack line, and adopt the technique of a non-matching grid interpolation between the fine and the coarse meshes. Numerical experiments demonstrate the effectiveness of the two-level domain decomposition preconditioners applied to elastic crack problems.

AMS subject classifications: 65M50, 65N50

Key words: Extended finite element method, domain decomposition, two-level preconditioners, elastic crack problem, non-matching grid.

1 Introduction

The extended finite element method (XFEM) proposed in 1999 [1,2,11] is a versatile technique for the computation of problems with discontinuities, singularities and localized

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deformations, etc. If the standard finite element method (FEM) is used for these problems, one often needs to use special meshes whose element edges coincide with the crack surface and nodes be placed on each side of the crack to allow material separations along the crack line. The construction of such meshes is quite difficult, in general. Moreover, mesh refinement is often needed in order to achieve optimal convergence, which increases the dofs remarkably. In contrast, in XFEM, the mesh is independent of the crack surface, which means structured meshes can be adopted. Furthermore, sometimes optimal convergence can be obtained by introducing a small number of additional basis functions into the approximation. These advantages of XFEM are particularly important for the modeling of materials with discontinuities.

Although XFEM is an attractive technique for modeling cracks, the algebraic system arising from XFEM is very ill-conditioned, see [3, 4]. For example, consider the linear elasticity problem, the condition number of the stiffness matrix in XFEM is $\mathcal{O}(h^{-4})$ (h is the mesh size), even if a single, non-polynomial enrichment function is used, while the condition number of the stiffness matrix in FEM is only $\mathcal{O}(h^{-2})$. Therefore, robust solvers are needed in XFEM. One way to reduce the condition number is to modify the enrichment functions. A stable generalized FEM (SGFEM) is proposed for one-dimensional problems in [3,4]. It shows that SGFEM is optimally convergent and has no blending elements, and the condition number is not worse than that of FEM. However, this idea can not be generalized to two-dimensional problems directly. In [14], the computational accuracy and the condition number of SGFEM are investigated for two-dimensional fracture mechanics problems. By adopting modified enrichment functions and linear Heaviside enrichments, this SGFEM yields accurate result and as the same time not deteriorating the condition number. Unfortunately, the condition number of this SGFEM depends on the location of the crack, and it is not really stable; see [19]. Using a local orthogonalized scheme, a strong SGFEM is constructed for the Poisson crack problem [19]. The optimal convergence is proved mathematically, and numerical experiments show that it is indeed stable. As a follow up to [19], a strong SGFEM for two-dimensional interface problems is proposed in [6], and they also proved mathematically that the convergence is optimal and the stiffness matrix is well-conditioned. As far as we know, there is no relevant literature about the strong SGFEM for elastic crack problems.

The other way to reduce the condition number of XFEM is through preconditioning. There are two types of XFEM, namely topological XFEM and geometric XFEM. For topological XFEM, there are several preconditioning techniques. In [12], a domain decomposition method is employed, and the condition number of the preconditioned system is close to that of FEM without any enrichments. In [10], a simple and efficient scheme is proposed for XFEM which only includes the Heaviside enrichment function. A special smoothed aggregation algebraic multigrid (AMG) preconditioner and an adaptive domain decomposition preconditioner are constructed in [15] and [18], respectively. In [5], a multiplicative Schwarz preconditioner is used, where the physical domain is decomposed into a "cracked" subdomain and several "healthy" subdomains. The "cracked" subproblem, which contains both the dofs of the Heaviside enrichment function and the