Parameter Identification in Uncertain Scalar Conservation Laws Discretized with the Discontinuous Stochastic Galerkin Scheme

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Abstract. We study an identification problem which estimates the parameters of the underlying random distribution for uncertain scalar conservation laws. The hyperbolic equations are discretized with the so-called discontinuous stochastic Galerkin method, i.e., using a spatial discontinuous Galerkin scheme and a Multielement stochastic Galerkin ansatz in the random space. We assume an uncertain flux or uncertain initial conditions and that a data set of an observed solution is given. The uncertainty is assumed to be uniformly distributed on an unknown interval and we focus on identifying the correct endpoints of this interval. The first-order optimality conditions from the discontinuous stochastic Galerkin discretization are computed on the time-continuous level. Then, we solve the resulting semi-discrete forward and backward schemes with the Runge-Kutta method. To illustrate the feasibility of the approach, we apply the method to a stochastic advection and a stochastic equation of Burgers' type. The results show that the method is able to identify the distribution parameters of the random variable in the uncertain differential equation even if discontinuities are present.

1 Introduction

Uncertainties play a role in many socio-economic, biological or physical phenomena which can be modelled with the help of partial differential equations (PDE) [4, 8, 12, 13, 19, 25, 31, 53]. In the recent years many approaches, e.g., methods based on Bayesian

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inversion, Monte Carlo algorithms or stochastic Galerkin schemes, were proposed to quantify the uncertainties in order to account for them in predictions and simulations [1,2,20,26,35,47,48,52,58,65].

In this article we focus on hyperbolic conservation laws having uncertainties in parameters which arise, i.e., due to measurement errors and thus have non-deterministic effects on the approximation of the deterministic problem. These uncertainties in the parameters can be modelled by random variables that follow an appropriate distribution type. In general, it is difficult to prescribe the exact distribution parameters if measurement errors are present. Therefore, our goal is to identify these distribution parameters from observed data that forms a solution to the uncertain conservation law. This yields the formulation of an optimization problem, whereas the hyperbolic partial differential equation constraint poses severe difficulties both at the continuous and discrete level since they typically form shocks even for smooth initial data when the flux function is non-linear.

Another difficulty is raised by the fact that the typical solution spaces for hyperbolic equations have no Hilbert space structure, therefore standard techniques for Optimal Control with PDE constraints are not applicable. We therefore pursue a discretization within the spatial and stochastic variable to formulate the identification problem in time-continuous form. On the ODE level we follow the approach 'first optimize, then discretize' motivated by the findings of [34]. This has the advantage that the state and the adjoint problem can be solved with different techniques leading to higher efficiency.

Discretizing the conservation law within the stochastic domain, we consider Uncertainty Quantification (UQ) methods [3,22,37,39,40,45,50] that aim to model the propagation of the uncertainty into the solution of uncertain equations. We distinguish between the so-called non-intrusive and intrusive schemes. The most widely known non-intrusive UQ method is (Multi-Level) Monte Carlo [13,24,30,41], which is based on statistical sampling methods that can easily be adopted to our problem setting but comes with potentially high costs due to the repeated application of finite volume schemes. Another approach is to employ a discretization in space which leads to a stochastic differential system. Here one could try to apply an parameter identification in the spirit of [46]. Within this article, we concentrate on an intrusive UQ method, namely the stochastic Galerkin (sG) scheme, that involves modifications of the finite volume solver. The method relies on the generalized Polynomial Chaos (gPC) expansion [1,14,60,63,64], thus expands the solution in the stochastic variable and projects it on the space spanned by a truncated orthonormal basis.

The biggest challenge of UQ methods for hyperbolic equations lies in the fact, that discontinuities in the physical space propagate into the solution manifold such that the polynomial expansion of discontinuous data yields huge oscillations [7,47,49]. Therefore, the authors of [61] introduced the so-called Multielement approach, where the random space is divided into disjoint elements in order to define local gPC approximations. Further developments of this method can be found in [54,59,62]. Similar to this ansatz, we apply a spatial discontinuous Galerkin discretization [15–17], where we expand the solution in