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Convolution Neural Network Shock Detector for Numerical Solution of Conservation Laws

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Abstract. We propose a universal discontinuity detector using convolution neural network (CNN) and apply it in conjunction of solving nonlinear conservation laws in both 1D and 2D. The CNN detector is trained offline with synthetic data. The training data are generated using randomly constructed piecewise functions, which are then processed using randomized linear advection solver to count for the cases of numerical errors in practice. The detector is then paired with high-order numerical solvers. In particular, we combined high-order WENO in troubled cells with high-order central difference in smooth region. Extensive numerical examples are presented. We observe that the proposed method produces notably sharper and cleaner signals near the discontinuities, when compared to other well known troubled cell detector methods.

AMS subject classifications: 35L65, 35L67, 65M06, 65M99

Key words: Deep neural network, convolution neural network, discontinuity detection, troubled cell, hybrid method, hyperbolic conservation laws.

1 Introduction

One of the challenges for solving nonlinear hyperbolic conservation laws is that even with smooth initial data, the solution may develop discontinuities in finite time. Without an appropriate treatment near the discontinuities, a high-order numerical scheme may generate spurious Gibbs oscillations and may even converge to an entropy-violating solution. The mesh cells containing low regularity of the solution are called "troubled cells." Capture of these cells is crucial in simulations, and its corresponding numerical techniques are referred to as troubled cell indicators or shock detectors in the literature.

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Various shock detection techniques have been developed and studied. Some are developed for finite volume or discontinuous Galerkin (DG) schemes and are used in the form of suitable limiters, see, for example, the minmod-based TVB limiter [4], the moment limiter of Biswas et al. [2] and its modification [3], the monotonicity-preserving limiter [32] and its modification [27], the KXRCF shock-detection technique by Krivodonova et al. [17], and the troubled cell indicator of Fu and Shu [8]. Some of the other shock detection techniques are aimed at improving the numerical performance when high-order essentially non-oscillatory (ENO) or weighted ENO (WENO) type approximations are used. See, for example, multi-resolution (MR) analysis of Harten [10], strong troubled cell indicator of Xu and Shu [37], etc. There have been extensive studies of these shock detection techniques in the literature, and we refer to [24] for a thorough numerical comparison in the context of DG methods and also [21] for hybrid finite difference methods.

Besides these traditional shock detection techniques, there are a few recent work on identifying troubled cells using artificial neural networks (ANNs). Compared with classical indicators, the neural network based indicators are usually free of problem-dependent parameters and are able to avoid certain mislabels, such as smooth extrema in minmod-based indicators. In [25], Ray and Hesthaven proposed to train a multi-layer perceptron, based on a supervised learning strategy, as a troubled cell indicator for DG methods. Its generalization to two-dimensional problems on unstructured grids was studied in [26]. In [33], the idea has also been pursued by Veiga and Abgrall with the study on transferred learning for adapting different methods or meshes. In [7], Feng et al. proposed a characteristic-featured shock wave indicator with one linear hidden layer. Their analysis shows that the indicator guarantees the only detection of discontinuities caused by characteristic curves compressing or intersecting. In [35], Wen et al. studied the combination of an ANN based troubled cell indicator and hybrid finite difference WENO methods.

Motivated by [34], our paper pursues the use of the convolution neural network (CNN) architecture for training the indicator. Widely used in image classification, CNN employs a series of convolution, pooling, and activation operations, followed by fully connected layers. The convolutional operations with kernels involved allow for parameter sharing and feature sharing, which to some degrees carry out invariance and generalizability properties when the CNN is well-trained. With those properties, a well-trained CNN is capable to reliably detect different types of discontinuities as demonstrated in [34]. Therefore, instead of specifying the local stencil as in some existing work, we use more global data in a subregion as the input, and rely on the CNN to extract the relevant information for determining troubled cells. Note that the key for a CNN detector to be successful is the training procedure, which often involves a large training set. Therefore, one of the main challenges in this project is to generate appropriate synthetic data to train our CNNs. We construct the training data in the finite dimensional numerical solution space which may exhibit some different structures near discontinuities, for instance, either smeared discontinuity or spurious oscillation may be observed. We start from piecewise smooth functions as the initial condition and apply numerical solvers on linear advection equations to evolve them for a few time steps to generate the training