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Numerical Simulations for Full History Recursive Multilevel Picard Approximations for Systems of High-Dimensional Partial Differential Equations

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Abstract. One of the most challenging issues in applied mathematics is to develop and analyze algorithms which are able to approximately compute solutions of high-dimensional nonlinear partial differential equations (PDEs). In particular, it is very hard to develop approximation algorithms which do not suffer under the curse of dimensionality in the sense that the number of computational operations needed by the algorithm to compute an approximation of accuracy $\varepsilon > 0$ grows at most polynomially in both the reciprocal $1/\varepsilon$ of the required accuracy and the dimension $d \in \mathbb{N}$ of the PDE. Recently, a new approximation method, the so-called *full history recursive multilevel Picard* (MLP) approximation method, has been introduced and, until today, this approximation scheme is the only approximation method in the scientific literature which has been proven to overcome the curse of dimensionality in the numerical approximation of semilinear PDEs with general time horizons. It is a key contribution of this article to extend the MLP approximation method to systems of semilinear PDEs and to numerically test it on several example PDEs. More specifically, we apply the proposed MLP approximation method in the case of Allen-Cahn PDEs, Sine-Gordon-type PDEs, systems of coupled semilinear heat PDEs, and semilinear Black-Scholes PDEs in up to 1000 dimensions. We also compare the performance of the proposed MLP approximation algorithm with a deep learning based approximation method from the scientific literature.

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1 Introduction

One of the most challenging issues in applied mathematics is to develop and analyze algorithms which are able to approximately compute solutions of high-dimensional nonlinear partial differential equations (PDEs). In particular, it is very hard to develop approximation algorithms which do not suffer under the curse of dimensionality in the sense that the number of computational operations needed by the algorithm to compute an approximation of accuracy $\varepsilon > 0$ grows at most polynomially in both the reciprocal $1/\varepsilon$ of the required accuracy and the dimension $d \in \mathbb{N}$ of the PDE. In the last four years, very significant progress has been made in this research area, where particularly the following two types of approximation methods have turned out to be very promising:

(I) Deep learning based approximation methods for PDEs; cf., e.g., [3–5, 9–12, 15–17, 19, 20, 24, 26, 28, 31, 36–40, 46, 48, 49, 52–63, 65, 66]

(II) Full history recursive multilevel Picard approximation methods for PDEs; cf., e.g., [6,7,22,23,29,41,43–45] (in the following we abbreviate *full history recursive multilevel Picard* by MLP)

Roughly speaking, deep learning based approximation methods for high-dimensional PDEs are often based on the idea

(Ia) to approximate the solution of the considered PDE through the solution of a suitable infinite dimensional stochastic optimization problem on an appropriate function space,

(Ib) to approximate some of the functions appearing in the infinite dimensional stochastic optimization problem by deep neural networks (DNNs) to obtain finite dimensional stochastic optimization problems, and

(Ic) to apply stochastic gradient descent type algorithms to the resulting finite dimensional stochastic optimization problems to approximately learn the optimal parameters of the involved DNNs.

MLP approximation methods have first been proposed in [22,43] and are, roughly speaking, based on the idea

(IIa) to reformulate the computational problem under consideration as a stochastic fixed point equation on a suitable function space with the fixed point of the fixed point equation being the solution of the computational problem,