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Learning to Discretize: Solving 1D Scalar Conservation Laws via Deep Reinforcement Learning

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Abstract. Conservation laws are considered to be fundamental laws of nature. It has broad applications in many fields, including physics, chemistry, biology, geology, and engineering. Solving the differential equations associated with conservation laws is a major branch in computational mathematics. The recent success of machine learning, especially deep learning in areas such as computer vision and natural language processing, has attracted a lot of attention from the community of computational mathematics and inspired many intriguing works in combining machine learning with traditional methods. In this paper, we are the first to view numerical PDE solvers as an MDP and to use (deep) RL to learn new solvers. As proof of concept, we focus on 1-dimensional scalar conservation laws. We deploy the machinery of deep reinforcement learning to train a policy network that can decide on how the numerical solutions should be approximated in a sequential and spatial-temporal adaptive manner. We will show that the problem of solving conservation laws can be naturally viewed as a sequential decision-making process, and the numerical schemes learned in such a way can easily enforce long-term accuracy. Furthermore, the learned policy network is carefully designed to determine a good local discrete approximation based on the current state of the solution, which essentially makes the proposed method a meta-learning approach. In other words, the proposed method is capable of learning how to discretize for a given situation mimicking human experts. Finally, we will provide details on how the policy network is trained, how well it performs compared with some state-of-the-art numerical solvers such as WENO schemes, and supervised learning based approach L3D and PINN, and how well it generalizes.

AMS subject classifications: 65M06, 68T05, 49N90
Key words: Conservation laws, deep reinforcement learning, finite difference approximation, WENO.

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1 Introduction

Conservation laws are considered to be one of the fundamental laws of nature, and has broad applications in multiple fields such as physics, chemistry, biology, geology, and engineering. For example, Burgers equation, a very classic partial differential equation (PDE) in conservation laws, has important applications in fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow.

Solving the differential equations associated with conservation laws has been a major branch of computational mathematics [18, 19], and a lot of effective methods have been proposed, from classic methods such as the upwind scheme, the Lax-Friedrichs scheme, to the advanced ones such as the ENO/WENO schemes [22, 30], the flux-limiter methods [14], and etc. In the past few decades, these traditional methods have been proven successful in solving conservation laws. Nonetheless, the design of some of the high-end methods heavily relies on expert knowledge and the coding of these methods can be a laborious process. To ease the usage and potentially improve these traditional algorithms, machine learning, especially deep learning, has been recently incorporated into this field. For example, the ENO scheme requires lots of 'if/else' logical judgments when used to solve complicated system of equations or high-dimensional equations. This very much resembles the old-fashioned expert systems. The recent trend in artificial intelligence (AI) is to replace the expert systems by the so-called 'connectionism', e.g., deep neural networks, which leads to the recent bloom of AI. Therefore, it is natural and potentially beneficial to introduce deep learning in traditional numerical solvers of conservation laws.

1.1 Related works

In the last few years, neural networks (NNs) have been applied to solving ODEs/PDEs or the associated inverse problems. These works can be roughly classified into two categories according to the way that the NN is used.

The first type of works propose to harness the representation power of NNs, and are irrelevant to the numerical discretization based methods. For example, in the pioneering works [2, 26, 27, 31, 34], NNs are used as new ansatz to approximate solutions of PDEs. It was later generalized by [33] to allow randomness in the solution which is trained using policy gradient. More recent works along this line include [6, 23, 25]. Besides, several works have focused on using NNs to establish direct mappings between the parameters of the PDEs (e.g. the coefficient field or the ground state energy) and their associated solutions [9,16,17,21]. Furthermore, [4,13] proposed a method to solve very high-dimensional PDEs by converting the PDE to a stochastic control problem and use NNs to approximate the gradient of the solution.

The second type of works, which target at using NNs to learn new numerical schemes, are closely related to our work. However, we note that these works mainly fall in the setting of supervised learning (SL). For example, [7] proposed to integrate NNs into high-order numerical solvers to predict artificial viscosity; [28] trained a multilayer perceptron