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High Order Conservative Semi-Lagrangian Scheme for the BGK Model of the Boltzmann Equation

Sebastiano Boscarino[1,*], Seung-Yeon Cho[1], Giovanni Russo[1] and Seok-Bae Yun[2]

[1] *Department of Mathematics and Computer Science, University of Catania, 95125 Catania, Italy.*
[2] *Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea.*

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Abstract. In this paper, we present a conservative semi-Lagrangian finite-difference scheme for the BGK model. Classical semi-Lagrangian finite difference schemes, coupled with an L-stable treatment of the collision term, allow large time steps, for all the range of Knudsen number [17, 27, 30]. Unfortunately, however, such schemes are not conservative. Lack of conservation is analyzed in detail, and two main sources are identified as its cause. First, when using classical continuous Maxwellian, conservation error is negligible only if velocity space is resolved with sufficiently large number of grid points. However, for a small number of grid points in velocity space such error is not negligible, because the parameters of the Maxwellian do not coincide with the discrete moments. Secondly, the non-linear reconstruction used to prevent oscillations destroys the translation invariance which is at the basis of the conservation properties of the scheme. As a consequence the schemes show a wrong shock speed in the limit of small Knudsen number. To treat the first problem and ensure machine precision conservation of mass, momentum and energy with a relatively small number of velocity grid points, we replace the continuous Maxwellian with the discrete Maxwellian introduced in [22]. The second problem is treated by implementing a conservative correction procedure based on the flux difference form as in [26]. In this way we can construct conservative semi-Lagrangian schemes which are Asymptotic Preserving (AP) for the underlying Euler limit, as the Knudsen number vanishes. The effectiveness of the proposed scheme is demonstrated by extensive numerical tests.

AMS subject classifications: 65L06, 65M25, 76P05

Key words: BGK model, Boltzmann equation, semi-Lagrangian scheme, conservative correction, discrete Maxwellian.

*Corresponding author. *Email addresses:* boscarino@dmi.unict.it (S. Boscarino), chosy89@skku.edu (S.-Y. Cho), russo@dmi.unict.it (G. Russo), sbyun01@skku.edu (S.-B. Yun)

1 Introduction

The dynamics of a non-ionized dilute gas at mesoscopic level is described by the celebrated Boltzmann equation [9]. The development of efficient numerical methods for its solution, however, constitutes a formidable challenge, due, among others, to the high dimensionality of the problem, the complicated structure of the collision operator, the need to preserve the collision invariants at a discrete level, and the stiffness issue arising when the Knudsen number is very small.

In view of this situation, Bhatnagar, Gross and Krook, in 1954, suggested a relaxation model of the Boltzmann equation, which now goes by the name of the BGK model [5]. This approximation preserves several important qualitative features of the original Boltzmann equation, such as conservation of mass, momentum and energy, H-theorem and relaxation to equilibrium, and is now widely used as a simplified alternative to the Boltzmann equation because it is much less expensive to treat at a numerical level.

Initial value problem for the BGK model on a periodic domain reads

$$\begin{aligned} &\frac{\partial f}{\partial t}+v\cdot\nabla_x f=\frac{1}{\kappa\tau_0}(\mathcal{M}(f)-f),\\ &f(x,v,0)=f_0(x,v). \end{aligned} \tag{1.1}$$

The velocity distribution function $f(x,v,t)$ represents the mass density of particles at point $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$ in phase space, at time $t>0$. The quantity $\tau=\kappa\tau_0$ represent the relaxation time. Here κ is the Knudsen number, defined as a ratio between the mean free path and a macroscopic characteristic length of the physical system. We assume it may change by several orders of magnitude, and in particular it may become extremely small. The time τ_0 expresses the dependence of the relaxation time on the deviation of temperature and density from the reference one. We assume such dependence is not very strong, and for simplicity we consider τ_0 to be constant in our treatment and analysis. By suitable non-dimensionalization of the problem we shall omit to write the term τ_0. The local Maxwellian $\mathcal{M}(f)$ is given by

$$\mathcal{M}(f)(x,v,t):=\frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^d}}\exp\left(-\frac{|v-U(x,t)|^2}{2T}\right),$$

where the macroscopic fields of local density $\rho(x,t)\in\mathbb{R}^+$, bulk velocity $U(x,t)\in\mathbb{R}^d$ and local temperature $T(x,t)\in\mathbb{R}^+$ are defined through the following relation:

$$(\rho(x,t),\rho(x,t)U(x,t),E(x,t))^T=\langle f\phi(v)\rangle, \tag{1.2}$$

where

$$\phi(v)=\left(1,v,\frac{1}{2}|v|^2\right)^T,\quad\text{and}\quad\langle g\rangle\equiv\int_{\mathbb{R}^d}g(v)dv.$$