Commun. Comput. Phys. Vol. **29**, No. 1, pp. 211-236
doi: 10.4208/cicp.OA-2020-0011 January 2021

Analysis and Application of Single Level, Multi-Level Monte Carlo and Quasi-Monte Carlo Finite Element Methods for Time-Dependent Maxwell's Equations with Random Inputs

Xiang Wang[1], Jichun Li[2,*] and Zhiwei Fang[2]

[1] *Department of Mathematical Sciences, Nanchang University, Nanchang, China.*
[2] *Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, Nevada 89154-4020, USA.*

Received 15 January 2020; Accepted (in revised version) 4 May 2020

Abstract. This article is devoted to three quadrature methods for the rapid solution of stochastic time-dependent Maxwell's equations with uncertain permittivity, permeability and initial conditions. We develop the mathematical analysis of the error estimate for single level Monte Carlo method, multi-level Monte Carlo method, and the quasi-Monte Carlo method. The theoretical results are supplemented by numerical experiments.

AMS subject classifications: 65N30, 35L15, 78-08

Key words: Maxwell's equations, uncertainty quantification, edge element.

1 Introduction

Consider the Maxwell's equations with random coefficients which are parameterized by a random vector $\boldsymbol{y}\in\mathbb{R}^n$:

$$\epsilon(\boldsymbol{x},\boldsymbol{y})\partial_t\boldsymbol{E}(\boldsymbol{x},t;\boldsymbol{y})=\nabla\times\boldsymbol{H}(\boldsymbol{x},t;\boldsymbol{y}), \tag{1.1}$$

$$\mu(\boldsymbol{x},\boldsymbol{y})\partial_t\boldsymbol{H}(\boldsymbol{x},t;\boldsymbol{y})=-\nabla\times\boldsymbol{E}(\boldsymbol{x},t;\boldsymbol{y}), \tag{1.2}$$

where ϵ is the permittivity, μ is the permeability, and $\boldsymbol{E}$ and $\boldsymbol{H}$ represent the electric and magnetic fields, respectively. We assume that the spatial variable $\boldsymbol{x}\in D\subseteq\mathbb{R}^3$, and the time variable $t\in(0,T]$. Here D is a bounded Lipschitz polyhedral domain with connected

*Corresponding author. *Email addresses:* `wangxiang49@ncu.edu.cn` (X. Wang), `jichun.li@unlv.edu` (J. Li), `fangz1@unlv.nevada.edu` (Z. Fang)

boundary ∂D. The curl operators are understood to operate on the spatial variables $\boldsymbol{x}$, and the parameter vector $\boldsymbol{y}=(y_1,y_2,\cdots,y_n)=(y_i)_{i=1}^n$ consists of n parameters y_i which are assumed to be independent and identically distributed (i.i.d.) on $[0,1]$, i.e.,

$$\boldsymbol{y}\in[0,1]^n:=U.$$

The probability measure for $\boldsymbol{y}$ is defined as $d\boldsymbol{y}=\Pi_{i=1}^n dy_i$.

To complete the problem, we assume that Eqs. (1.1)-(1.2) satisfy the perfect conducting (PEC) boundary condition:

$$\mathbf{n}\times\boldsymbol{E}(\boldsymbol{x},t;\boldsymbol{y})=0 \quad \forall \boldsymbol{x}\in\partial D,\ \forall t\in(0,T] \text{ and } \forall \boldsymbol{y}\in U, \tag{1.3}$$

where $\mathbf{n}$ is the outward unit normal vector on ∂D. Furthermore, we assume that the Maxwell's equations (1.1)-(1.2) satisfy the following initial conditions:

$$\boldsymbol{E}(\boldsymbol{x},0;\boldsymbol{y})=\boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{H}(\boldsymbol{x},0;\boldsymbol{y})=\boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}), \tag{1.4}$$

where $\boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y})$ are some given functions.

In the past two decades, the study of uncertainty quantification (UQ) got great attentions across different disciplines of sciences and engineering as detailed in recent review articles [19, 27, 36, 38] and monographs [16, 33, 41, 42, 47, 50]. Uncertainty quantification plays an important part in electromagnetic material design. For example, [46] presented a computational stochastic methodology for generating and optimizing random metamaterial configurations. Compared to many excellent numerical analysis papers published for stochastic elliptic problems (e.g., [1,2,8,49]), stochastic parabolic equations [48] and stochastic hyperbolic or wave equations [24, 26, 35, 43, 45, 52], elastic waves scattering in random media [3, 15], stochastic porous media flow [13], radiative transfer equations with uncertain coefficients [51], the mathematical literature on UQ for Maxwell's equations is less developed. In 2006, Chauviere et al. [6] developed both the stochastic Galerkin method and stochastic collocation method for the time-dependent Maxwell's equations with uncertainties caused by the physical materials, the source wave and the physical domain. In 2015, Benner and Schneider [5] described several techniques for the time-harmonic Maxwell's equations by using stochastic collocation method. In 2016, Römer et al. [37] discussed a stochastic nonlinear magnetostatic problem solved by the stochastic collocation method. In 2018, Kamilis and Polydorides [25] considered an UQ problem for the low-frequency, time-harmonic Maxwell's equations with lognormal random conductivity. Also Jerez et al. [23] and Hao et al. [20] investigated the time-harmonic Maxwell's equations with random interfaces. In 2019, Chen et al. [7] analyzed a semi-implicit Euler scheme for discretizing the stochastic Maxwell's equations with multiplicative Itô noise, and derived the mean-square convergence. Recently, the authors carried out the error analysis of stochastic collocation method [30, 31] and stochastic Galerkin method [14] for time-dependent Maxwell's equations.

In this paper, the uncertain permittivity and permeability ϵ and μ in (1.1)-(1.2) are assumed to depend on both the spatial variable $\boldsymbol{x}$ and the parameter $\boldsymbol{y}$, and they are