

Discrete-Velocity Vector-BGK Models Based Numerical Methods for the Incompressible Navier-Stokes Equations

Jin Zhao*

Division of Applied and Computational Mathematics, Beijing Computational Science Research Center, Beijing 100193, P.R. China.

Received 4 November 2019; Accepted (in revised version) 4 February 2020

Abstract. In this paper, we propose a class of numerical methods based on discrete-velocity vector-BGK models for the incompressible Navier-Stokes equations. By analyzing a splitting method with Maxwell iteration, we show that the usual lattice Boltzmann discretization of the vector-BGK models provides a good numerical scheme. Moreover, we establish the stability of the numerical scheme. The stability and second-order accuracy of the scheme are validated through numerical simulations of the two-dimensional Taylor-Green vortex flows. Further numerical tests are conducted to exhibit some potential advantages of the vector-BGK models, which can be regarded as competitive alternatives of the scalar-BGK models.

AMS subject classifications: 65M06, 65M12, 35Q30, 76P05

Key words: Vector-BGK models, incompressible Navier-Stokes equations, Maxwell iteration, weighted L^2 -stability.

1 Introduction

The Boltzmann equation for colliding hard spheres [11, 12, 24] reads as

$$\partial_t f + v \cdot \nabla f = Q(f). \quad (1.1)$$

Here $f = f(t, x, v)$ denotes the particle distribution function at space-velocity-time $(t, x, v) \in [0, \infty) \times \mathbb{R}^D \times \mathbb{R}^D$ and $Q(f)$ is a collision term having the property

$$\int_{\mathbb{R}^D} Q(f) \varphi(v) dv = 0 \quad (1.2)$$

*Corresponding author. *Email address:* zjin@csrc.ac.cn (J. Zhao)

for $\varphi(v) = 1, v, |v|^2$, giving the conservation laws of mass, momentum and energy. In this paper, the collision term is taken as

$$Q(f) = \frac{M(f) - f}{\tau_0} \tag{1.3}$$

with $M(f)$ the local Maxwellian computed by the velocity-moments of f and the positive parameter τ_0 the relaxation time. This collision operator is usually referred to as the BGK model and was introduced by Bhatnagar et al. in [3].

In this paper, we consider the discrete-velocity vector-BGK model

$$\partial_t f_i + \frac{\lambda}{\epsilon} e_i \cdot \nabla f_i = \frac{M_i(w) - f_i}{\tau_0 \epsilon^2}, \quad i = 1, 2, \dots, N. \tag{1.4}$$

Here $f_i = f_i(t, x)$ is the i -th unknown vector-valued function taking values in \mathbb{R}^{D+1} , $e_i = (e_{i1}, \dots, e_{iD})$ is the i -th discrete velocity included in a finite and symmetric discrete set $\mathbb{V} := \{e_i\} = \{-e_i\}$, λ is a positive number, $\epsilon > 0$ is a small parameter,

$$w = \sum_{i=1}^N f_i,$$

and $M_i = M_i(w)$ is a given vector-valued function of w . A detailed description of the model (1.4) is given in Section 3 (see also [1, 8, 28]) and specific examples for $M(w)$ can be found in Section 5. In previous works [4, 5, 28], it was shown that the discrete-velocity vector-BGK model can be regarded as an approximation, when $\epsilon > 0$ is small, to the incompressible Navier-Stokes equations

$$\begin{cases} \nabla \cdot u = 0, \\ \partial_t u + \nabla \cdot (u \otimes u) + \nabla \Phi = \nu \Delta u, \end{cases} \tag{1.5}$$

where $u = u(t, x)$, $\Phi = \Phi(t, x)$ and $\nu > 0$ denote the fluid velocity, pressure and viscosity, respectively. This means that approximate solutions of the incompressible Navier-Stokes equations can be obtained by solving the vector-BGK models.

In order to numerically solve the vector-BGK models, we may split the models into the collision and transport steps, which can be both solved analytically. For the splitting method obtained thus, we exploit the Maxwell iteration to show that the method may not be valid for high Reynolds number flows. Based on this, we realize that the usual lattice Boltzmann (LB) discretization (for example, [12, 15, 24]) can provide a better numerical scheme:

$$f_i(t_n + \Delta t, x_j + \Delta t c e_i) = f_i(t_n, x_j) + Q_i(f(t_n, x_j)), \quad i = 1, 2, \dots, N. \tag{1.6}$$

Here $c = \lambda/\epsilon$, Δt denotes the time step, $t_n = n\Delta t$ and node $x_j \in \Omega$ with the node set Ω generated by the N discrete velocities. When each $f_i = f_i(t, x)$ is a scalar function, it is