

Asymptotic Structure of Cosmological Burgers Flows in One and Two Space Dimensions: A Numerical Study

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Abstract. We study the cosmological Burgers model, as we call it, which is a nonlinear hyperbolic balance law (in one and two spatial variables) posed on an expanding or contracting background. We design a finite volume scheme that is fourth-order in time and second-order in space, and allows us to compute weak solutions containing shock waves. Our main contribution is the study of the asymptotic structure of the solutions as the time variable approaches infinity (in the expanding case) or zero (in the contracting case). We discover that a saddle competition is taking place which involves, on one hand, the geometrical effects of expanding or contracting nature and, on the other hand, the nonlinear interactions between shock waves.

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1 Introduction

Balance law of interest. We investigate numerically the global dynamics of a compressible fluid containing shock waves and evolving on a curved background spacetime of a contracting or expanding type. Motivated by the (inviscid) Burgers equation that has played such a central role in standard fluid dynamics, we consider here its relativistic version

$$av_t + f(v)_x + g(v)_y = a_t h(v), \quad (x, y) \in [0, L]^2, \quad \epsilon v \in (-1, 1), \quad (1.1)$$

which we refer to as the **cosmological Burgers model**. This equation provides a simple setup for designing and testing shock-capturing schemes in a curved spacetime background and investigating the asymptotic behavior of weak solutions.

In (1.1), the unknown is a function $v = v(t, x, y) \in (-1/\epsilon, 1/\epsilon)$ representing the main velocity component of a fluid vector field, and $1/\epsilon$ represents the speed of light. The fluxes $f = f(v)$ and $g = g(v)$ and the source function $h = h(v)$ are given smooth functions. We formulate the evolution on the domain $[0, L]^2$ with vanishing boundary conditions. A typical choice of flux and source functions is

$$f(v) = g(v) = \frac{1}{2}v^2, \quad h(v) = -v(1 - \epsilon^2 v^2), \quad (1.2)$$

which allows us to recover the standard Burgers equation by taking the limit $a \rightarrow 1$ and $\epsilon \rightarrow 0$.

Geometric background of interest. The function $a = a(t) > 0$ describes a geometric background of contracting or expanding type. Shock wave solutions to nonlinear hyperbolic equations such as (1.1) are only defined in the *forward time direction* and, since the equation is singular at $t = 0$, it is natural to distinguish between two initial value problems corresponding to the following range of the time variable:

- In the range $t \in [t_0, +\infty)$, the background is assumed to be expanding toward the future in the sense that $a(t)$ increases monotonically to $+\infty$ and data are prescribed at some $t_0 > 0$.
- In the range $t \in [t_0, 0)$, the background is assumed to be contracting toward the future in the sense that $a(t)$ decreases monotonically to 0 and data are prescribed at some $t_0 < 0$.

A typical choice is the function $a(t) = a_0(t/t_0)^\alpha$, which we can normalize by taking $a_0 = 1$ and $t_0 = \pm 1$, in which $\alpha \in (0, 1)$ represents the rate of contraction or expansion of the background:

$$a(t) = |t|^\alpha. \quad (1.3)$$

Our model is motivated from the full Euler system posed on the so-called FLRW background (after Friedmann–Lemaître–Robertson–Walker) describing a homogeneous and isotropic cosmology, for which a typical exponent is $\alpha = 2/3$.

Earlier works. Structure-preserving methods for non-homogeneous conservation laws have been proposed by many authors in the past twenty years. Let us, for instance, refer to Russo [21, 22] (and the references cited therein) for models of shallow water, and to Chertock et al. [8] about models taking gravity effects into account. In Beljadid et al. [2, 3], a finite volume method was designed for geometric conservation laws posed on the sphere. In all of these papers, the difficulty lies in the preservation of the effect induced by source-terms of gravity or geometric type. See also [4, 10].

We recall that the inviscid Burgers equation has played a central role in the development of shock-capturing schemes for problems arising in non-relativistic fluid dynamics.