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The Immersed Interface Method for Non-Smooth Rigid Objects in Incompressible Viscous Flows

Yang Liu and Sheng Xu*

Department of Mathematics, Southern Methodist University, Dallas, TX 75275, USA.

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Abstract. In the immersed interface method, an object in a flow is formulated as a singular force, and jump conditions caused by the singular force are incorporated into numerical schemes to compute the flow. Previous development of the method considered only smooth objects. We here extend the method to handle non-smooth rigid objects with sharp corners in 2D incompressible viscous flows. We represent the boundary of an object as a polygonal curve moving through a fixed Cartesian grid. We compute necessary jump conditions to achieve boundary condition capturing on the object. We incorporate the jump conditions into finite difference schemes to solve the flow on the Cartesian grid. The accuracy, efficiency and robustness of our method are tested using canonical flow problems. The results demonstrate that the method has second-order accuracy for the velocity and first-order accuracy for the pressure in the infinity norm, and is extremely efficient and robust to simulate flows around non-smooth complex objects.

AMS subject classifications: 76T99, 35J05, 65N06

Key words: The immersed interface method, boundary condition capturing, non-smooth complex rigid objects, jump conditions.

1 Introduction

Many numerical methods have been proposed to solve various interface problems [5, 6, 10]. Among them, the immersed interface method has become a general framework for numerically solving differential equations with interfaces [5]. Since it was first proposed by LeVeque and Li [3, 4] to improve the first-order accuracy of Peskin's immersed boundary method [10], it has been developed and studied by many researchers for various interface problems. A long list of references can be found in [5]. In a series of papers [17, 19–26], we have developed the method to simulate incompressible viscous flows with moving objects in both 2D and 3D and two-fluid flows in 2D.

*Corresponding author. *Email addresses:* sxu@smu.edu (S. Xu), yang@smu.edu (Y. Liu)

When employed to simulate flows with immersed objects, the immersed interface method inherits the same mathematical formulation from Peskin's immersed boundary method [9, 10] – that is, the boundary of an immersed object in a flow is formulated as a singular force in the Navier-Stokes equations. In Peskin's immersed boundary method, the singular force is regularized by a compactly supported smooth function, leading to a smeared boundary and first-order accuracy. In the immersed interface method, jump conditions induced by the singular force are derived or computed with a boundary representation and then directly incorporated into numerical schemes, leading to a sharp boundary and second-order or higher accuracy. However, in previous implementations of the immersed interface method, only smooth interfaces and objects were considered. In the current paper, we extend the method to handle non-smooth complex rigid objects with sharp corners in 2D incompressible viscous flows. The two main contributions of the paper are (1) computation of necessary jump conditions and (2) boundary condition capturing using panel representation (instead of global parametrization) of a boundary.

1.1 Formulation for the immersed interface method

For a rigid object (multiple objects can be included in the similar manner) in a 2D incompressible viscous flow (Fig. 1), the primitive-variable formulation reads

$$\frac{\partial \vec{u}}{\partial t}+\nabla\cdot(\vec{u}\otimes\vec{u})=-\nabla p+\frac{1}{Re}\Delta\vec{u}+\int_\Gamma \vec{f}\left(\vec{X}(s),t\right)\delta\left(\vec{x}-\vec{X}(s)\right)\mathrm{d}s+\vec{b}, \tag{1.1}$$

$$\Delta p=s_p+\nabla\cdot\left(\int_\Gamma \vec{f}\left(\vec{X}(s),t\right)\delta\left(\vec{x}-\vec{X}(s)\right)\mathrm{d}s+\vec{b}\right), \tag{1.2}$$

where $\vec{u}=(u,v)$, p and Re are the velocity, pressure and Reynolds number of the flow, respectively, Γ is the closed boundary of the object, s is the arc length along Γ, $\vec{f}$ is the density of the singular force distributed along Γ via the 2D Dirac delta function $\delta(\vec{x}-\vec{X})$, $\vec{x}=(x,y)$ is the position vector for the domain, $\vec{X}=(X,Y)$ is the position vector for Γ, and $\vec{b}$ is a discontinuous body force that is applied only inside Γ (and is zero outside). The no-slip and no-penetration boundary conditions on the object can be effectively achieved by the singular force and the discontinuous body force using the boundary condition capturing technique [19, 21] (see Section 2). The pressure Poisson equation, Eq. (1.2), is obtained by taking the divergence of the momentum equation, Eq. (1.1). The non-homogeneous term s_p in the pressure Poisson equation is

$$s_p=-\left(\frac{\partial D}{\partial t}+\nabla\cdot(2\vec{u}D)-\frac{1}{Re}\Delta D\right)+2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right), \tag{1.3}$$

where $D=\nabla\cdot\vec{u}$ is the divergence of the velocity, which is zero theoretically due to the incompressibility of the flow. We keep the terms consisting of D above in computation to better enforce the incompressibility condition numerically.