Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0112

Vol. **29**, No. 2, pp. 534-570
February 2021

A Sufficient and Necessary Condition of the Existence of WENO-Like Linear Combination for Finite Difference Schemes

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Received 7 July 2019; Accepted (in revised version) 9 January 2020

Abstract. In the finite difference WENO (weighted essentially non-oscillatory) method, the final scheme on the whole stencil was constructed by linear combinations of highest order accurate schemes on sub-stencils, all of which share the same total count of grid points. The linear combination method which the original WENO applied was generalized to arbitrary positive-integer-order derivative on an arbitrary (uniform or non-uniform) mesh, still applying finite difference method. The possibility of expressing the final scheme on the whole stencil as a linear combination of highest order accurate schemes on WENO-like sub-stencils was investigated. The main results include: (a) the highest order of accuracy a finite difference scheme can achieve and (b) a sufficient and necessary condition that the linear combination exists. This is a sufficient and necessary condition for all finite difference schemes in a set (rather than a specific finite difference scheme) to have WENO-like linear combinations. After the proofs of the results, some remarks on the WENO schemes and TENO (targeted essentially non-oscillatory) schemes were given.

AMS subject classifications: 65J05, 76M20

Key words: Finite difference, WENO, sufficient and necessary condition, proof.

1 Introduction

In nonlinear hyperbolic conservation systems (e.g. Euler equations of inviscid compressible flow), the solution may develop discontinuities even if the initial value is smooth,

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due to the intrinsic nonlinearity. Several techniques were developed to tackle discontinuities in such problems. Some of the techniques represent the discontinuities as actual discontinuities, e.g. shock fitting [1, 2], jump recovery [3], and subcell resolution [4]. Different from the techniques listed above, the shock capturing method [5, 6] artificially spreads a discontinuity in several (typically less than 10) cells, turning the discontinuity into a large gradient zone. Such methods automatically ”capture” shock waves without special treatments, thus the algorithms of such an approach are simpler. The foundation of shock capturing methods is artificial viscosity, which was developed by Von Neumann and Richtmyer [7, 8]. At present, the artificial viscosity is often introduced implicitly via the flux vector splitting (FVS) method [9].

One of the shock capturing schemes is the essentially non-oscillatory scheme [10–12], the basic idea of which is to select the ”smoothest” stencil to perform calculations. Based on the essentially non-oscillatory (ENO) scheme, weighted essentially non-oscillatory (WENO) scheme [13] was developed. The WENO scheme can be considered as an improved version of the ENO scheme. When shock is detected, the WENO scheme degenerates to the ENO scheme. If shock is not present, a higher order accurate flux will be calculated using all the fluxes calculated in the ENO scheme. Such a procedure improves the scheme’s resolution in smooth zones.

Later in 1996, a generic framework to design finite difference WENO schemes [14] was published. The designing framework was a great success, yet still some issues require further considerations. Firstly, the framework actually asked for a uniform mesh to directly apply the WENO schemes. When applying the WENO schemes on a non-uniform smoothly varying mesh, one must perform a transformation to make the mesh uniform [15]. Secondly, in practice, it was noted that the order of accuracy may drop near critical points [16]. Thirdly, WENO schemes typically were unnecessarily dissipative at small scales [17]. Fourthly, stability problems may occur for very-high-order WENO schemes [18].

For the first issue that the framework was designed for uniform or smoothly varying mesh, research showed that it is feasible to apply the WENO schemes on a non-uniform mesh directly, using the finite volume method [20, 21]. If one applies finite difference schemes directly on a non-uniform mesh, the order of accuracy cannot be greater than 2 with the prerequisites that (a) the derivatives are evaluated in a conservative form, i.e.

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_j = \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} \tag{1.1}$$

and (b) the numerical flux $f_{j+1/2}$ expressed as

$$f_{j+1/2} = \sum_{i=r}^{s} c_{ij} f_{j+i} \tag{1.2}$$

with the coefficients c_{ij} independent of mesh sizes [15].