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One-Dimensional Blood Flow with Discontinuous Properties and Transport: Mathematical Analysis and Numerical Schemes

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Abstract. In this paper we consider the one-dimensional blood flow model with discontinuous mechanical and geometrical properties, as well as passive scalar transport, proposed in [E.F. Toro and A. Siviglia. Flow in collapsible tubes with discontinuous mechanical properties: mathematical model and exact solutions. *Communications in Computational Physics*. 13(2), 361-385, 2013], completing the mathematical analysis by providing new propositions and new proofs of relations valid across different waves. Next we consider a first order DOT Riemann solver, proposing an integration path that incorporates the passive scalar and proving the well-balanced properties of the resulting numerical scheme for stationary solutions. Finally we describe a novel and simple well-balanced, second order, non-linear numerical scheme to solve the equations under study; by using suitable test problems for which exact solutions are available, we assess the well-balanced properties of the scheme, its capacity to provide accurate solutions in challenging flow conditions and its accuracy.

AMS subject classifications: 76Z05, 35L02, 65M08

Key words: Blood flows, Riemann problem, wave relations, finite volume method, well-balancing.

1 Introduction

Blood circulation is fundamental to human physiology, having as main functions the transport of substances and the regulation of temperature. Modelling such processes in single vessels or in networks of vessels allows to study the effect of different stimuli on

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blood flow from a systemic point of view. In turn, these effects are related to changes in the geometry and mechanical properties of vessels. Arterial circulation has been the focus of attention in the past decades for researchers looking at pathological states of the cardiovascular system and its role played in other diseases. However, nowadays clinical research is looking into the potential role played by the venous system in the development and clinical course of neurodegenerative diseases such as *Idiophatic Parkinson's disease* [1], *Ménière disease* [2–4], *multiple sclerosis* [5–8]. As pointed out in a special issue on venous system modelling published in 1969 by the journal *IEEE Transactions on Biomedical Engineering*, main aspects of the venous system posing modelling challenges are collapsibility of veins and the relevance of external forces such as gravity and external pressure to venous flow. Recently, researchers have addressed the description of flow in collapsible tubes [9–12] and related numerical applications to rather simple problems [13–15]. Moreover, some interesting works have been published regarding tube laws for veins [16], and the construction of global multiscale mathematical models for the human circulation [17, 18]. Some of these works derived in the study of neurological diseases [17, 19–21].

One-dimensional (1D) blood flow models [10] have been extensively used to study wave propagation phenomena in arteries [22–27]. More recently, their use has been extended to the venous circulation [17,18,20]. 1D models satisfactorily describe wave propagation phenomena in vascular networks and have considerably lower computational cost than full 3D Fluid Structure Interaction (FSI) models [28, 29]. These models allow to investigate physical mechanisms underlying changes in pressure and flow waveforms that are produced by cardiovascular disease. 1D blood flow models form the basis of wave analysis tools for extracting clinically relevant information from waveform measurements, for example, separation of waves into forward and backward-travelling components [30] and wave intensity analysis [31, 32]. As it will be shown in Section 2, the underlying equations for this model constitute a non-linear hyperbolic system of partial differential equations. The unknowns of this system are cross-sectional area, flow rate and pressure, thus a closure condition is needed: the so-called tube law. Such closure relations differ depending on the vascular district of application, with rather simplified relations for arteries, whereas for veins more sophisticated and challenging relations are needed, especially if vessel collapse is addressed.

In this paper we focus our attention on physical situations of medical interest in which certain properties that characterize compliant vessels change rapidly in space, for example after the insertion of stents in arteries or in veins due to a surgical procedure. A common pathology in the human circulatory system is the presence of atherosclerotic plaques that can cause restrictions of the arterial lumen called stenoses. In the most severe cases stenoses may hinder, or even stop, the flow of blood. One of the available techniques to treat this problem is the implantation of a stent (an expandable metal mesh) into the affected region which has the purpose of returning the artery lumen to approximately its original shape. Whenever possible, this procedure is preferred to more invasive ones, such as surgical by-pass. Nevertheless, besides other effects, the presence