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DL-PDE: Deep-Learning Based Data-Driven Discovery of Partial Differential Equations from Discrete and Noisy Data

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Abstract. In recent years, data-driven methods have been developed to learn dynamical systems and partial differential equations (PDE). The goal of such work is to discover unknown physics and corresponding equations. However, prior to achieving this goal, major challenges remain to be resolved, including learning PDE under noisy data and limited discrete data. To overcome these challenges, in this work, a deep-learning based data-driven method, called DL-PDE, is developed to discover the governing PDEs of underlying physical processes. The DL-PDE method combines deep learning via neural networks and data-driven discovery of PDE via sparse regressions. In the DL-PDE, a neural network is first trained, then a large amount of meta-data is generated, and the required derivatives are calculated by automatic differentiation. Finally, the form of PDE is discovered by sparse regression. The proposed method is tested with physical processes, governed by the diffusion equation, the convection-diffusion equation, the Burgers equation, and the Korteweg-de Vries (KdV) equation, for proof-of-concept and applications in real-world engineering settings. The proposed method achieves satisfactory results when data are noisy and limited.

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1 Introduction

As data acquisition and storage ability has increased, data-driven methods have been utilized for solving various problems in different fields [1–4]. In recent years, data-driven discovery of governing equations of physical problems has attracted much attention. Instead of building models from physical laws, the goal of such an approach is to discover unknown physics and corresponding equations directly from limited observation data. Substantial progress has been made in terms of proof-of-concept and preliminary applications. Among these investigations, sparse regression methods are frequently used techniques, which show promise for discovering the governing partial differential equations (PDEs) of various problems. Using sparse regression aims to identify a small number of terms that constitute a governing equation from a predefined large candidate library, and a parsimonious model can usually be obtained. Sparse identification of nonlinear dynamics (SINDy), sequential threshold ridge regression (STRidge), and Lasso are proposed to identify PDE from data [5–7]. Since then, a large body of extant literature has investigated data-driven discovery of governing equations using sparse regression [4, 8–19]. Despite the numerous successes achieved with sparse regression-based methods, major challenges remain when faced with noisy data and limited data. Since numerical approximation of derivatives is requisite in these methods, the results may be unstable and ill-conditioned when handling noisy data [20]. Total variation, polynomial interpolation, and the integral form are utilized to handle noisy data [5,6,13]. However, these strategies can only lessen the difficulties associated with noisy data to a certain extent.

Besides the sparse regression method, other techniques, such as Gaussian process and neural networks, are also used for performing data-driven discovery of governing equations. For example, Raissi et al. [21] proposed a framework that utilizes the Gaussian process to discover governing equations. In their proposed framework, parameters of the differential operator are turned into hyper-parameters of some covariance functions and are learned by the maximum likelihood method. Meanwhile, the physics-informed neural network (PINN) is presented for solving forward and inverse problems of PDE [22]. In the PINN, by adding a PDE constraint term in the loss function, in addition to the data match term, the accuracy of the results can be improved and the coefficients of the PDE terms can be learned. Avoiding the numerical approximation of derivatives, both the Gaussian process-based method and the neural network-based method require less data and are less sensitive to data noise [21,22]. However, in the above-mentioned works, the PDE of the considered problem is supposed to have a known structure and only the coefficients of the PDE terms are learned from data, which limits its application for PDE discovery. To overcome this limitation, Raissi [23] modified the PINN by introducing two neural networks for approximating the unknown solution, as well as the unknown PDE. Even though this modification enables the PINN to solve problems with unknown PDE structures, the learned neural network approximation of the unknown PDE is a black box, and thus lacks interpretability. Long et al. [24] employed a convolutional neural network to identify the form of the unknown PDE. However, parsimony of the results