Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2020-0047

Vol. **29**, No. 3, pp. 747-766
March 2021

A Nonlinear Finite Volume Scheme Preserving Maximum Principle for Diffusion Equations

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Received 13 March 2020; Accepted (in revised version) 15 July 2020

Abstract. In this paper we propose a new nonlinear cell-centered finite volume scheme on general polygonal meshes for two dimensional anisotropic diffusion problems, which preserves discrete maximum principle (DMP). The scheme is based on the so-called diamond scheme with a nonlinear treatment on its tangential flux to obtain a local maximum principle (LMP) structure. It is well-known that existing DMP preserving diffusion schemes suffer from the fact that auxiliary unknowns should be presented as a convex combination of primary unknowns. In this paper, to get rid of this constraint a nonlinearization strategy is introduced and it requires only a second-order accurate approximation for auxiliary unknowns. Numerical results show that this scheme has second-order accuracy, preserves maximum and minimum for solutions and is conservative.

AMS subject classifications: 65N08, 65M22

Key words: Maximum principle, finite volume scheme, diffusion equation.

1 Introduction

Diffusion processes are ubiquitous in applications such as petroleum engineering, porous media flow and energy transport in inertial confinement fusion. The maximum principle or minimum principle is a significant property of diffusion equations [1], which means the physical solution of a diffusion equation is bounded below or (and) above by known

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data. Simulation of such diffusion problems requires robust and highly accurate numerical methods with respect to physical property such as local conservation and maximum principle especially on skewed meshes. In [2–5], many second-order linear schemes on skewed meshes are introduced, but none of them satisfies discrete maximum principle (DMP), which may lead to spurious oscillation or non-physical solutions. This deficiency motivated researchers exploring more robust discrete schemes satisfying DMP for diffusion equations.

Since a linear scheme with second-order accuracy always fails to satisfy DMP [6, 7], nonlinear schemes are considered. A second-order maximum principle preserving finite volume method for steady convection-diffusion problems is proposed in [8]. However, it has a restriction on geometric shape of the meshes. A specific structure called local maximum principle (LMP) structure that ensures the discrete local maximum and minimum principles is introduced in [9], which provides a criterion of designing such schemes. LMP structure leads to that the coefficient matrix of scheme has non-positive off-diagonal elements with non-negative row sums. Followed by LMP structure, a nonlinear correction for general finite volume scheme to satisfy DMP is presented in [10], and some convergence results are proved there. It corrects the sum of all edges fluxes on a cell to give LMP structure and seems over-corrected so that only first-order accuracy is obtained in some numerical results presented there, even though the original scheme is a second-order one. Another approach of designing schemes satisfying DMP is to construct one-side flux with LMP structure on each edge, where a convex decomposition for co-normal vector is used. To do this, some auxiliary points, such as edge midpoints or harmonic averaging points or cell-vertices are introduced (see [11–14] for details). All the values at auxiliary points have to be expressed as a convex combination of cell-centered ones around, which either needs impose some severe restrictions on meshes and diffusion coefficient regularity or results in complicated interpolation algorithms especially on the place in the case of diffusion coefficient being discontinuous even for nonlinear vertex interpolation method like [15]. For Galerkin approximations, some nonlinear stabilized terms are introduced in [16,17] for convection-diffusion equation and Laplace equation to satisfy DMP and some theoretical results are presented there under finite element framework. Some diffusion schemes preserving maximum principle on non-orthogonal quadrilateral meshes are constructed in [18] by introducing simple flux limiters, moreover it is proved that under some appropriate conditions, the coercivity, the boundness and the convergence of the discrete solution in the discrete H^1 norm hold. Discrete duality finite volume schemes (DDFVS) with positivity-preserving are proposed in [19,20] by decoupling cell-centered unknowns and vertex unknowns. However, the idea of DDFVS is not easy to applied to DMP preserving schemes since DMP preserving schemes require more hash condition on fluxes.

We present a new nonlinear finite volume scheme with LMP structure for anisotropic diffusion problem with discontinuous coefficients on general star-shaped polygonal meshes. We start from the linear second-order one-side flux in [2] and [13], which is different from the flux in [8, 9, 11], and then introduce a nonlinearization strategy, i.e., an