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A Nodal Finite Element Method for a Thermally Coupled Eddy-Current Problem with Moving Conductors

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Abstract. This paper aims to design and analyze a solution method for a time-dependent, nonlinear and thermally coupled eddy-current problem with a moving conductor on hyper-velocity. We transform the problem into an equivalent coupled system and use the nodal finite element discretization (in space) and the implicit Euler method (in time) for the coupled system. The resulting discrete coupled system is decoupled and implicitly solved by a time step-length iteration method and the Picard iteration. We numerically and theoretically prove that the finite element approximations have the optimal error estimates and both the two iteration methods possess the linear convergence. For the proposed method, numerical stability and accuracy of the approximations can be held even for coarser mesh partitions and larger time steps. We also construct a preconditioner for the discrete operator defined by the linearized bilinear form and show that this preconditioner is uniformly effective. Numerical experiments are done to confirm the theoretical results and illustrate that the proposed method is well behaved in large-scale numerical simulations.

AMS subject classifications: 65N30, 65N55

Key words: Thermally coupled eddy-current problem, finite element method, time step-length iteration, Picard iteration, optimal error estimates, preconditioner.

1 Introduction

Let Ω be a bounded and connected Lipschitz domain in three dimensions, and Ω be decomposed into a union of two non-overlapping subdomains Ω_c and Ω_n. In applications,

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Ω_c and Ω_n denote a conducting area and a nonconducting area, respectively. The domain Ω_c is half-surrounded by the domain Ω_n.

The behavior of an electromagnetic system obeys five basic integral laws: Amperes Circuital Law, Faradays Induction Law, Law of Source Free Magnetic Flux, Gauss Law, and Law of Charge Conservation [1]. They are described as the following:

$$\oint_{\partial S} \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}\cdot\mathbf{n}ds + \frac{\partial}{\partial t}\int_S \epsilon\mathbf{E}\cdot\mathbf{n}ds, \tag{1.1a}$$

$$\oint_{\partial S} \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot\mathbf{n}ds, \tag{1.1b}$$

$$\int_{\partial D} \mathbf{B}\cdot\mathbf{n}ds = 0, \tag{1.1c}$$

$$\int_{\partial D} \epsilon\mathbf{E}\cdot\mathbf{n}ds = \int_D \rho d\mathbf{x}, \tag{1.1d}$$

$$\int_{\partial D} \mathbf{J}\cdot\mathbf{n}ds + \frac{\partial}{\partial t}\int_D \rho d\mathbf{x} = 0, \tag{1.1e}$$

where $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic flux density, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, ϵ is the permittivity and ρ is the volume charge density. Here $D\subseteq\Omega$ can be chosen as any simply-connected subdomain of Ω, S can be chosen as any piecewise smooth directed surface contained in Ω and $\mathbf{n}$ denotes the unit out normal vector.

Since (1.1d) can be deduced from (1.1a) and (1.1e) [1], we do not consider the law (1.1d) (whose differential form is $\nabla\cdot(\epsilon\mathbf{E})=\rho$) in the considered model. In the underlying applications, the displacement current can be neglected, i.e., $\frac{\partial}{\partial t}\int_D \rho d\mathbf{x}\equiv 0$ in (1.1e). Then by virtue of the Gauss divergence theorem and Stokes theorem, the remaining four laws (1.1a), (1.1b), (1.1c) and (1.1e) lead to the following differential form of Maxwell's equations defined in Ω (see [12]):

$$\begin{cases} \nabla\times\mathbf{H}=\mathbf{J} & \text{in } \Omega_c, \\ \nabla\times\mathbf{H}=0 & \text{in } \Omega_n, \\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} & \text{in } \Omega_c, \\ \nabla\cdot\mathbf{B}=0 & \text{in } \Omega, \end{cases} \quad \text{with} \quad \begin{cases} \mathbf{B}=\mu\mathbf{H} & \text{in } \Omega, \\ \mathbf{J}=\sigma(T)\mathbf{E} & \text{in } \Omega_c. \end{cases} \tag{1.2}$$

The magnetic permeability μ is set to be a constant, and σ is the electrical conductivity depending on the temperature T, which needs to be solved by a thermodynamic partial differential equation (referring to (2.7) in Section 2).

The above coupled model is used to simulate multi-physical fields in the electrical energy transducers, such as motors, transformers and the electromagnetic rail launcher (ERL), which can convert electrical energy into mechanical energy and vice versa (see [8,19]). In applications, the field $\mathbf{E}$ may be expressed as the sum of two terms:

$$\mathbf{E}=\mathbf{E}_c+\boldsymbol{\vartheta}\times\mathbf{B} \quad \text{in } \Omega_c, \tag{1.3}$$