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Structure-Preserving Numerical Methods for Stochastic Poisson Systems

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Abstract. We propose a numerical integration methodology for stochastic Poisson systems (SPSs) of arbitrary dimensions and multiple noises with different Hamiltonians in diffusion coefficients, which can provide numerical schemes preserving both the Poisson structure and the Casimir functions of the SPSs, based on the Darboux-Lie theorem. We first transform the SPSs to their canonical form, the generalized stochastic Hamiltonian systems (SHSs), via canonical coordinate transformations found by solving certain PDEs defined by the Poisson brackets of the SPSs. An α-generating function approach with $\alpha \in [0,1]$ is then constructed and used to create symplectic schemes for the SHSs, which are then transformed back by the inverse coordinate transformation to become stochastic Poisson integrators of the original SPSs. Numerical tests on a three-dimensional stochastic rigid body system illustrate the efficiency of the proposed methods.

AMS subject classifications: 60H35, 60H15, 65C30, 60H10, 65D30

Key words: Stochastic Poisson systems, Poisson structure, Casimir functions, Poisson integrators, symplectic integrators, generating functions, stochastic rigid body system.

1 Introduction

Poisson systems form a class of important mechanical systems whose long history dates back to the 19th century [10, 17, 28]. As a generalization of the Hamiltonian systems which are defined on even-dimensional symplectic manifolds, the poisson systems possess similar but extended structural properties, and can be defined on Poisson manifolds of arbitrary dimensions. They have a large scope of applications, such as in astronomy,

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robotics, fluid mechanics, electrodynamics, quantum mechanics, nonlinear waves, and so on [42]. Unlike Hamiltonian systems where plenty literatures are available on their numerical approximations, there have not been as many studies on numerical simulations for the Poisson systems. One of the main challenges for numerical approximations of the Poisson systems is that such approximations depend on the concrete structure matrix, which makes it difficult to establish general methodologies [12, 17].

Symplectic methods for Hamiltonian systems have been developed during the last decades [12,17,35]. They find applications in many fields where Hamiltonian systems appear, and are proved to be much superior than non-symplectic methods in long time simulation, due to their ability of preserving the symplectic structure of the original systems (see e.g. [5, 6, 19]). Structure-preserving algorithms of a broader sense are then aroused which seek for preservation of more structural conservation law in numerical discretization, such as energy, momentum, etc. (see e.g. [9, 16, 20]). The Poisson structure is an extension of the symplectic structure to arbitrary-dimension and variable structure matrices, and is reduced to the symplectic structure when the structure matrices degenerate to the even-dimensional symplectic matrix J. It is an intrinsic structure of the Poisson systems. However, it has been observed that, symplectic methods in general do not preserve the Poisson structure [12,17,37]. Therefore, there is a need to develop Poisson integrators which can inherit the Poisson structure of the Poisson systems. Such attempts have been made for deterministic cases in e.g. [8, 15, 21, 25, 30, 36, 39, 42] etc.

In recent years, there arise some numerical studies on certain special stochastic Poisson systems (SPSs). [7] proposed a class of energy-preserving numerical methods for stochastic Poisson systems where the drift and diffusion coefficients vary by a constant among each other, and systems where the diffusion coefficients are multiplications of constant skew-symmetric matrices and the gradient of the same Hamiltonian with that of the drift part. These methods are proved to preserve quadratic Casimir functions as well. [26] constructed a class of explicit parametric stochastic Runge-Kutta methods with truncated random variables for stochastic Poisson systems of the form as in [7], and showed that these methods are energy-preserving for suitable parameters, and can be of any prescribed convergence orders. For stochastic Poisson systems of even dimensions and invertible structure matrices, [18] investigated structure-preserving Runge-Kutta and partitioned Runge-Kutta type methods. Up to now, however, to the best of our knowledge, there is still no general setting of stochastic Poisson systems, which can cover cases of arbitrary dimensions and multiple noises with different Hamiltonians in diffusion coefficients of the systems, nor systematic structure-preserving numerical analysis for them.

In this paper, we propose a class of numerical methods for general stochastic Poisson systems. By appropriate coordinate transformations, we rewrite the SPSs into their canonical forms, which are generalized stochastic Hamiltonian systems (SHSs). Then we apply a stochastic α-generating function approach to construct symplectic schemes for the resulted SHSs, and transform the symplectic schemes back to numerical schemes for the SPSs afterwards. The so-proposed methods are shown to preserve the Poisson structure and the Casimir functions of the SPSs. Suitable coordinate transformations are found