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High-Order Runge-Kutta Discontinuous Galerkin Methods with a New Type of Multi-Resolution WENO Limiters on Tetrahedral Meshes

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Abstract. In this paper, the second-order and third-order Runge-Kutta discontinuous Galerkin (RKDG) methods with multi-resolution weighted essentially non-oscillatory (WENO) limiters are proposed on tetrahedral meshes. The multi-resolution WENO limiter is an extension of a finite volume multi-resolution WENO scheme developed in [81], which serves as a limiter for RKDG methods on tetrahedral meshes. This new WENO limiter uses information of the DG solution essentially only within the troubled cell itself which is identified by a new modified version of the original KXRCF indicator [42], to build a sequence of hierarchical L^2 projection polynomials from zeroth degree to the second or third degree of the DG solution. The second-order and third-order RKDG methods with the associated multi-resolution WENO limiters are developed as examples for general high-order RKDG methods, which could maintain the original order of accuracy in smooth regions and keep essentially non-oscillatory property near strong discontinuities by gradually degrading from the optimal order to the first order. The linear weights inside the procedure of the new multi-resolution WENO limiters can be set as any positive numbers on the condition that they sum to one. A series of polynomials of different degrees within the troubled cell itself are applied in a WENO fashion to modify the DG solutions in the troubled cell on tetrahedral meshes. These new WENO limiters are very simple to construct, and can be easily implemented to arbitrary high-order accuracy on tetrahedral meshes. Such spatial reconstruction methodology improves the robustness in the simulation on the same compact spatial stencil of the original DG methods on tetrahedral meshes. Extensive one-dimensional (run as three-dimensional problems on tetrahedral meshes) and three-dimensional tests are performed to demonstrate the good performance of the RKDG methods with new multi-resolution WENO limiters.

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1 Introduction

In this paper, three-dimensional hyperbolic conservation laws

$$\begin{cases} u_t+f(u)_x+g(u)_y+r(u)_z=0, \\ u(x,y,z,0)=u_0(x,y,z), \end{cases} \tag{1.1}$$

are considered and the Runge-Kutta discontinuous Galerkin (RKDG) methods [13–15, 17] with new multi-resolution WENO limiters are applied to solve (1.1) on tetrahedral meshes. The DG methods are applied to discretize the spatial variables and explicit, nonlinearly stable high-order Runge-Kutta methods [12, 18, 38, 48, 64, 66] are adopted to discretize the temporal variable. The main objective of this paper is to design new second-order and third-order spatial limiting procedures to obtain uniform accuracy in smooth regions and obtain sharp and non-oscillatory shock transitions in non-smooth regions for high-order RKDG methods. This new methodology can be applied to design high-order WENO limiting procedures for any high-order RKDG methods on tetrahedral meshes, however we will use only second-order and third-order cases in this paper as examples.

Let us first review the history of the development of discontinuous Galerkin (DG) methods. In 1973, Reed and Hill [62] designed the first DG method in the framework of neutron transport. Due to its desirable properties, many developed DG methods were also used in atmospheric science with an extensive list of references [2, 29–31, 55–57]. The reconstruction operator [21, 22] was applied at the beginning of each time step in the computation to increase the formal order of accuracy of high-order DG methods. Lagrangian DG methods were proposed for the first time in [49, 53, 68–70]. A Taylor basis was used in [40] for the development of a discontinuous Galerkin spectral finite element method. More recently, a new novel weighted Runge-Kutta discontinuous Galerkin method [37] is proposed for solving three-dimensional acoustic and elastic wave and reconstructed discontinuous Galerkin (rDG) method [51, 52] is presented for solving diffusion equations. Other new contributions to design high-order DG methods can also be found in [10, 11, 45]. But if problems are not smooth enough, the associated numerical solution would have spurious oscillations near strong shocks or contact discontinuities and could result in the appearances of nonlinear instability in non-smooth regions. One possible methodology to suppress such spurious oscillations is to apply nonlinear limiters to the RKDG methods. A major development of the DG method with a classical *minmod* type total variation bounded (TVB) limiter was carried out by Cockburn et al. in a series of papers [13–17] to solve nonlinear time dependent hyperbolic conservation laws