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A Mixed Finite Element Scheme for Quad-Curl Source and Eigenvalue Problems

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Abstract. The quad-curl problem arises in the resistive magnetohydrodynamics (MHD) and the electromagnetic interior transmission problem. In this paper we study a new mixed finite element scheme using Nédélec's edge elements to approximate both the solution and its curl for quad-curl problem on Lipschitz polyhedral domains. We impose element-wise stabilization instead of stabilization along mesh interfaces. Thus our scheme can be implemented as easy as standard Nédélec's methods for Maxwell's equations. Via a discrete energy norm stability due to element-wise stabilization, we prove optimal convergence under a low regularity condition. We also extend the mixed finite element scheme to the quad-curl eigenvalue problem and provide corresponding convergence analysis based on that of source problem. Numerical examples are provided to show the viability and accuracy of the proposed method for quad-curl source problem.

AMS subject classifications: 65M60, 65N30

Key words: Quad-curl problem, mixed finite element scheme, error estimates, eigenvalue problem.

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1 Introduction

The quad-curl term is an essential part in the magnetic induction equation of magnetohydrodynamics (MHD) with hyperresistivity [3, 8] and fourth-order electromagnetic transmission eigenvalue problem [6,7,17]. Let Ω be a Lipschitz bounded polyhedral domain in $\mathbb{R}^3$ and $\boldsymbol{f}\in[L^2(\Omega)]^3$ with $\nabla\cdot\boldsymbol{f}=0$. The quad-curl model problem can be described as follows:

$$\begin{cases} (\nabla\times)^4\boldsymbol{u}+\nabla p=\boldsymbol{f}, & \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u}=0, & \text{in } \Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=0, \quad (\nabla\times\boldsymbol{u})\times\boldsymbol{n}=0, & \text{on } \partial\Omega,\\ p=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $(\nabla\times)^4=(\nabla\times\nabla\times\nabla\times\nabla\times)$, p is the Lagrange multiplier and $\boldsymbol{n}$ is the unit normal of the boundary pointing towards the outside of Ω.

A natural variational formulation of (1.1) is to find $(\boldsymbol{u},p)\in H_0^2(\text{curl},\Omega)\times H_0^1(\Omega)$ such that

$$\begin{cases} (\nabla\times\nabla\times\boldsymbol{u},\nabla\times\nabla\times\boldsymbol{v})+(\nabla p,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}), & \forall\boldsymbol{v}\in H_0^2(\text{curl},\Omega),\\ (\boldsymbol{u},\nabla s)=0, & \forall s\in H_0^1(\Omega). \end{cases} \tag{1.2}$$

Here, $H_0^2(\text{curl},\Omega)=\{\boldsymbol{v}\in H_0(\text{curl},\Omega):\nabla\times\boldsymbol{v}\in H_0(\text{curl},\Omega)\}$ (see (2.3)), and $H(\text{div}^0,\Omega)=\{\boldsymbol{v}\in H(\text{div},\Omega):\nabla\cdot\boldsymbol{v}=0\}$. Obviously, $\nabla\cdot\boldsymbol{f}=0$ implies $p=0$. Though the variational formulation (1.2) is simple, it is not straightforward to design curl-curl finite element subspaces of $H_0^2(\text{curl},\Omega)$, if we want to avoid the phenomenon of wrong numerical approximation towards non-physical solutions like the Maxwell equations [11]. Very recently, Zhang et al. [25] developed a curl-curl element for studying the two-dimensional quad-curl problem. However, it is not obvious how to generalize the element in [25] for three dimensional domain in a practical way.

Several numerical methods have been developed to avoid using curl-curl finite element spaces. In [27], a nonconforming finite element based on a modified Morley-type element was proposed for the quad-curl problem under the assumption that $\boldsymbol{u}\in[H^4(\Omega)]^3$. A discontinuous Galerkin type method (DG) [15] uses the high order Nédélec's edge elements with stabilization for the jump of curl of the numerical solution along mesh interfaces. In this method, the regularity requirements were assumed as $\boldsymbol{u}\in[H^2(\Omega)]^3$ and $\nabla\times\boldsymbol{u}\in[H^2(\Omega)]^3$. Besides, the frequently used mixed finite element methods were also considered to handle the quad-curl problem. In [21], Sun developed a mixed finite element method in the Ciarlet and Raviart (C-R) scheme for the quad-curl and the corresponding eigenvalue problem assuming that $\boldsymbol{u}\in[H^3(\Omega)]^3$ and $\nabla\times\boldsymbol{u}\in[H^3(\Omega)]^3$. Some other mixed schemes and corresponding error analysis were also proposed in [24, 26]. In [17], the Maxwell transmission eigenvalue problem with the quad-curl operator was solved by the Ciarlet and Raviart method. In this method, a lower order curl conforming edge elements of Nédélec was adopted to compute the eigenvalues of the fourth order transmission eigenvalue problem. Later, Brenner et al. [4] studied the two-dimensional