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A Decoupled Energy Stable Adaptive Finite Element Method for Cahn–Hilliard–Navier–Stokes Equations

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Abstract. In this paper, we propose, analyze, and numerically validate an adaptive finite element method for the Cahn–Hilliard–Navier–Stokes equations. The adaptive method is based on a linear, decoupled scheme introduced by Shen and Yang [30]. An unconditionally energy stable discrete law for the modified energy is shown for the fully discrete scheme. A superconvergent cluster recovery based a posteriori error estimations are constructed for both the phase field variable and velocity field function, respectively. Based on the proposed space and time discretization error estimators, a time-space adaptive algorithm is designed for numerical approximation of the Cahn–Hilliard–Navier–Stokes equations. Numerical experiments are presented to illustrate the reliability and efficiency of the proposed error estimators and the corresponding adaptive algorithm.

AMS subject classifications: 65N15, 65N30, 65N50.

Key words: Cahn–Hilliard equation, Navier–Stokes equation, energy stability, adaptive, SCR.

1 Introduction

In this paper, we consider the adaptive finite element method for solving the following Cahn–Hilliard–Navier–Stokes (CHNS) equation [30]:

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$$\begin{cases} \partial_t\phi+\nabla\cdot(\boldsymbol{u}\phi)-\gamma\Delta w=0, & \text{in } \Omega\times[0,T], \quad (1.1a)\\ w+\lambda\left(\Delta\phi-\frac{1}{\varepsilon^2}f(\phi)\right)=0, & \text{in } \Omega\times[0,T], \quad (1.1b)\\ \partial_t\boldsymbol{u}-\mu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p+\phi\nabla w=0, & \text{in } \Omega\times[0,T], \quad (1.1c)\\ \nabla\cdot\boldsymbol{u}=0, & \text{in } \Omega\times[0,T], \quad (1.1d)\\ \boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0,\ \phi(\cdot,0)=\phi_0, & \text{in } \Omega\times\{t=0\}, \quad (1.1e)\\ \boldsymbol{u}|_{\partial\Omega}=0,\ \frac{\partial\phi}{\partial\boldsymbol{n}}|_{\partial\Omega}=0,\ \frac{\partial w}{\partial\boldsymbol{n}}|_{\partial\Omega}=0, & \quad (1.1f)\end{cases}$$

where $\Omega\subseteq\mathbb{R}^d$, $\boldsymbol{u}$ and p represent the velocity field and the pressure, respectively. ϕ means the phase field variable, where $\phi=\pm1$ corresponds to two different fluids and w is the chemical potential. The nonlinear term $f(\phi)=F'(\phi)=(\phi^3-\phi)$ with the double well free energy $F(\phi)=\frac{1}{4}(\phi^2-1)^2$, and the parameter $\varepsilon>0$, which usually represents the interface width, is a small parameter compared with the characteristic length of the laboratory scale. The parameter γ is a mobility constant related to the relaxation time scale, μ denotes the viscosity, and λ represents the magnitude of the mixing energy.

By taking the inner product of (1.1a) with $-w$, (1.1b) with $\partial_t\phi$, (1.1c) with $\boldsymbol{u}$, respectively, and after some algebraic manipulations, we have the following energy dissipative law:

$$\frac{d}{dt}E(\phi,\boldsymbol{u})=-\int_\Omega\left(\mu|\nabla\boldsymbol{u}|^2+\gamma|\nabla w|^2\right)dx\leq0, \quad (1.2)$$

with the total energy

$$E(\phi,\boldsymbol{u})=\int_\Omega\left(\frac{1}{2}|\boldsymbol{u}|^2+\frac{\lambda}{2}|\nabla\phi|^2+\frac{\lambda}{\varepsilon^2}F(\phi)\right)dx. \quad (1.3)$$

The phase field model CHNS equation (1.1a)-(1.1e), which is valid for fluids of matched density, is introduced to describe the dynamics of two phase, incompressible, and macroscopically immiscible Newtonian fluids with matched density [18,27]. In the past decades, the CHNS equation and its various modified forms have been applied in addressing a variety of situations, such as the solidification process of liquid metal alloys [9], the simulation of bubble dynamics [1]. For more physical background and the application of the CHNS equations, we refer to [25,35] and the references therein.

There are several challenges to construct efficient and easy-to-implement numerical methods for the CHNS equations. One of the theoretical and numerical difficulty associated with CHNS equations is the high order derivative involved in the Cahn–Hilliard equation. For the discretization of higher order derivative by finite element methods, it is natural to use the C^1 conforming finite element methods, which require the basis functions and their derivatives are continuous. But the C^1 finite element is rarely used in practice for their involving too many degrees of freedom and implementation complexity. An alternative choice is the mixed methods which rewrite the fourth order phase field